Industrial Organization - Solution to the Final Exam

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Jérôme MATHIS (LEDa)

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Exercise 1. Bertrand Competition Between AI Platforms (10 pts).

a) (1 pt) Firm A maximizes its profit, given by:

$$\max_{p_A \geq 0} \pi_A(p_A, p_B) = \max_{p_A \geq 0} (p_A - m) D_A = \max_{p_A \geq 0} (p_A - m)(a - b p_A + c p_B)$$

F.O.C. :

$$\frac{\partial \left(\pi_A(p_A, p_B)\right)}{\partial p_A} = 0 \iff p_A^*(p_B) = \frac{a + c p_B + bm}{2b}$$

S.O.C. is satisfied : $\frac{\partial^2 (\pi_A(p_A,p_B))}{\partial p_A^2} = -2 < 0$, so F.O.C. is sufficient. Similarly, for firm B, we obtain

$$p_B^*(p_A) = \frac{a + c p_A + bm}{2b}$$

b) (2 pts) The pure strategy Nash equilibrium in price (p_A^N, p_B^N) is given by:

$$p_A^*(p_B^*(p_A^N)) = p_A^N; \text{ and } p_B^*(p_A^*(p_B^N)) = p_B^N \tag{1}$$

which yields to

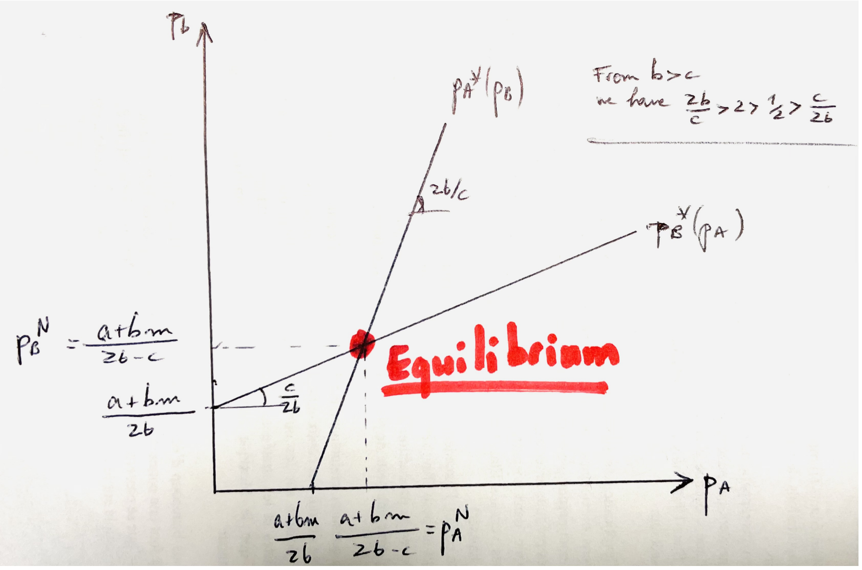
$$\frac{a + c(\frac{a + c p_A^N + bm}{2b}) + bm}{2b} = p_A^N \iff (2b - \frac{c^2}{2b}) p_A^N = a + c(\frac{a + bm}{2b}) + bm \iff (4b^2 - c^2) p_A^N = (a + bm)(2b + c)$$

That is

$$p_A^N = \frac{a + bm}{2b - c}; \text{ and } p_B^N = \frac{a + bm}{2b - c}.$$

The pure strategy Nash equilibrium in price is then symmetric and equals to $(p_A^N, p_B^N) = \left(\frac{a+bm}{2b-c}, \frac{a+bm}{2b-c}\right)$.

c) (2 pts) The graphical representation of this price equilibrium in the (p_A, p_B) space is:



d) (1 pt) The equilibrium prices clearly increases with:

- the baseline demand parameter a;
- the cross-price substitutability parameter c;[1] and
- the marginal cost m.
- From $\frac{\partial(p_A^N)}{\partial b} = -\frac{2a+mc}{(2b-c)^2} < 0$, the equilibrium prices decreases with the own-price sensitivity b.

e) (2 pts) The equilibrium quantities D_A^N and D_B^N at the Bertrand–Nash prices are given by

$$\begin{aligned} D_A^N &= a - bp_A^N + cp_B^N = a + (c-b)p_A^N = D_B^N \\ &= a + (c-b)\frac{a+bm}{2b-c} = \frac{b(a+m(c-b))}{2b-c} \end{aligned}$$

The resulting equilibrium profits are

$$\pi_A^N = (p_A^N - m)D_A^N = (\frac{a+bm}{2b-c} - m)\frac{b(a-m(b-c))}{2b-c} = b\left(\frac{a-m(b-c)}{2b-c}\right)^2 = \pi_B^N$$

The equilibrium profits clearly increase with the baseline demand parameter a; and, from $b > c$, decreases with the marginal cost m.

f) (1 pt) When the platforms collude and set a symmetric price, p^c, they maximize joint profit:

$$\begin{aligned} \Pi(p) &= \max_{p\geq 0} \pi_A(p,p) + \pi_B(p,p) = \max_{p\geq 0}(p-m)(D_A + D_B) \\ &= \max_{p\geq 0} 2(p-m)(a - p(b-c)) = \max_{p\geq 0} 2(-(b-c)p^2 + p(a + m(b-c) - am) \end{aligned}$$

F.O.C. :

$$\frac{\partial\left(\Pi(p)\right)}{\partial p} = 0 \iff p^c = \frac{a + m(b-c)}{2(b-c)}$$

S.O.C. is satisfied : $\frac{\partial^2(\Pi(p))}{\partial p^2} = -4(b-c) < 0$, so F.O.C. is sufficient. From $(p-m) = \frac{a-m(b-c)}{2(b-c)}$ and $(a - p(b-c)) = a - m(b-c)$, the collusive profit is

$$\Pi^c = \frac{(a - m(b-c))^2}{b-c}$$

g) (1 pt) When $a = 100$, $b = 2$, $c = 1$, and $m = 10$ we obtain

$$p_A^N = p_B^N = \frac{a+bm}{2b-c} = 40;\ p^c = \frac{a+m(b-c)}{2(b-c)} = 55;\ \pi_A^N + \pi_B^N = 3600;\ \Pi^c = \frac{(a-m(b-c))^2}{b-c} = 4050$$

Hence, compared to the Bertrand–Nash solutions, both the collusive price and aggregated profits are higher.

Exercise 2. Collusion in a quantity oligopoly

a) **(1 pt)** In the stage game, the highest aggregate profit that the firms can be led to share corresponds to the monopoly profit $\pi^m := max_{Q\in[0,1]}\pi(Q)$, where Q denotes the aggregate output, and market profit $\pi(Q)$ writes as $Qp(Q) - cQ$. The formation of this profit requires coordination of all the firms in the market to obtain a total monopoly production $Q^m = \sum_{i=1}^N q_i$. From $p = 1 - Q$ if $Q \leq 1$ and 0 otherwise, we have

$$\pi^m = \max_{Q\in[0,1]} Q(1-Q) - cQ$$

F.O.C :

$$\frac{\partial\left(-Q^2 + Q\left(1-c\right)\right)}{\partial Q} = 0 \iff Q^m = \frac{1-c}{2}$$

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[1] With differentiated Bertrand competition, higher cross-price substitutability makes prices more strategic complements: when one platform raises its price, the other optimally raises its own. Because undercutting becomes less profitable as products become closer substitutes, both firms end up setting higher equilibrium prices. We then have higher equilibrium prices than when models differ strongly. This matches classic differentiated-Bertrand models (Singh, N., & Vives, X. (1984). Price and quantity competition in a differentiated duopoly. The Rand journal of economics, 546-554.).

S.O.C is satisfied : $\frac{\partial^2\left(-Q^2+Q(1-c)\right)}{\partial Q^2} = -2 < 0$, so F.O.C is sufficient. The monopoly profit is thus worth

$$\pi^m = Q^m(1 - Q^m - c) = \left(\frac{1-c}{2}\right)^2.$$

b) (2 pts) This stage game has a unique Nash equilibrium payoff. Indeed, faced with a production level Q_{-i} of the competitors, firm i's best response writes as $q_i^*\left(Q_{-i}\right) \in \operatorname{argmax}_{\mathrm{q_i}\in[0,1]} q_i(1-(Q_{-i}+q_i)-c)$.

F.O.C :

$$\frac{\partial\left(q_i(1-(Q_{-i}+q_i-c)\right)}{\partial q_i} = 0 \iff -2q_i + 1 - c - Q_{-i} = 0 \iff q_i^*\left(Q_{-i}\right) = \frac{1-c-Q_{-i}}{2} \qquad (1)$$

S.O.C. is satisfied: $\frac{\partial^2 (q_i(1-(Q_{-i}+q_i)-c))}{\partial q_i^2} = -2 < 0$, so F.O.C is sufficient. The profit is then written

$$\pi_i\left(q_i^*\left(Q_{-i}\right), Q_{-i}\right) = q_i^*\left(Q_{-i}\right)\left(1-\left(Q_{-i}+q_i^*\left(Q_{-i}\right)\right)-c\right) = \left(\frac{1-c-Q_{-i}}{2}\right)\left(1-\left(Q_{-i}+\left(\frac{1-c-Q_{-i}}{2}\right)\right)-c\right)$$

so,

$$\pi_i\left(q_i^*\left(Q_{-i}\right), Q_{-i}\right) = \left(\frac{1-c-Q_{-i}}{2}\right)^2 = q_i^*\left(Q_{-i}\right)^2 \qquad (2)$$

At Cournot-Nash equilibrium, condition (1) is satisfied for any firm and by summing over $i \in N$ we obtain

$$\sum_{i=1}^{n} q_i^*\left(Q_{-i}\right) = Q^C \iff \sum_{i=1}^{n}\left(\frac{1-c-Q_{-i}}{2}\right) = Q^C \iff n\left(\frac{1-c}{2}\right) = Q^C + \sum_{i=1}^{n}\left(\frac{Q_{-i}^C}{2}\right) = Q^C + \sum_{i=1}^{n}\left(\frac{Q^C-q_i^C}{2}\right)$$

$$\iff n\left(\frac{1-c}{2}\right) = \frac{2Q^C + nQ^C - Q^C}{2} = \frac{(n+1)Q^C}{2}$$

Hence the aggregate Cournot output level: $Q^C = \frac{n(1-c)}{n+1}$. From (1) we have

$$q_i^*\left(Q_{-i}^C\right) = \frac{1-c-Q_{-i}^C}{2} = \frac{1-c-\left(Q^C - q_i^*\left(Q_{-i}^C\right)\right)}{2}$$

so

$$q_i^*\left(Q_{-i}^C\right) = 1 - c - Q^C = 1 - c - \frac{n\left(1-c\right)}{n+1} = \frac{1-c}{n+1} := q_i^C$$

We deduce that the Nash equilibrium of the stage game is symmetric: $q_i^C = q_j^C\ \forall i,j \in N$. The corresponding individual profit is written:

$$\pi_i^C = q_i^C\left(1-c-Q^C\right) = \left(\frac{1-c}{n+1}\right)\left(1-c-\frac{n\left(1-c\right)}{n+1}\right) = \left(\frac{1-c}{n+1}\right)^2 = \left(q_i^C\right)^2$$

which is consistent with (2). From $Q^C = \frac{n(1-c)}{n+1} < 1$, the market price is $p^C = 1 - Q^C = \frac{1+nc}{n+1}$.

c) (1 pt) When competition in quantities is repeated a finite number of times, the unique subgame perfect Nash equilibrium consists of the repetition of the Nash equilibrium of the stage game. No coalition is sustainable at the equilibrium of the repeated interaction. At the last period, there is no punishment threat and each player plays his stage game best-response. In the period before, the players know what will happen in the last period, so the current period becomes the "last meaningful period", and each player plays his stage game best-response as well. The same reasoning then applies at every period by backward induction.

d) (1 pt) In the infinitely repeated game, to sustain full collusion, define the following grim-trigger strategy for each firm $i \in N$:

- in period $t = 1$, produce the collusive quantity $q_i^m \equiv \frac{Q^C}{n} = \frac{1-c}{2n}$;

- at any period $t > 1$, produce q_i^m if every firm j has produced the quantity $q_j^m = \frac{1-c}{2n}$ in every previous period; otherwise, produce the Cournot quantity $q_i^C = \frac{1-c}{n+1}$ forever.

e) (2 pts) When each firm produces $q_i^m := \frac{Q^m}{n} = \frac{1-c}{2n}$. The individual profit is then written

$$\pi_i^m := q_i^m\left(1-c-Q^m\right) = (1-c)^2 4n$$

Clearly, $\frac{\pi_i^C}{\pi_i^m} = \frac{\left(\frac{1-c}{n+1}\right)^2}{\frac{(1-c)^2}{4n}} = \frac{4n}{(n+1)^2} < 1$ for all $n > 1$. So, $\pi_i^m > \pi_i^C$ for all $n > 1$, and (absent deviation incentives) firms strictly prefer collusion to Cournot.

Let us characterize the value of the discount factor δ beyond which the grim-trigger strategic profile of the previous question is an equilibrium. When all the firms form a cartel to share the monopoly profit equally, the profit of each firm is written:

$$\frac{\pi^m}{n}\sum_{t=0}^{T}\delta^t = \frac{\pi^m}{n}\frac{1-\delta^{T+1}}{1-\delta} \underset{T\to+\infty}{\longrightarrow} \frac{\pi^m}{n}\frac{1}{1-\delta}$$

The average profit is therefore written:

$$\frac{\pi^m}{n}\left(1-\delta\right)\sum_{t=0}^{T}\delta^t = \frac{\pi^m}{n}\left(1-\delta^{T+1}\right) \underset{T\to+\infty}{\longrightarrow} \frac{\pi^m}{n}$$

In accordance with answer **c)**, in the following we consider the infinite case ($T=+\infty$). By deviating for the first time at a given stage, a firm obtains at most at this stage:

$$\pi_i(q_i^*(Q_{-i}^m), Q_{-i}^m)$$

where $q_i^*(Q_{-i}^m)$ designates firm i's best response to the monopoly production of $(n-1)$ other firms $Q_{-i}^m = \frac{1-c}{2n}(n-1)$.

From (1) we have

$$q_i^*(Q_{-i}^m) = \frac{1-c-Q_{-i}^m}{2} = \frac{1-c-\frac{1-c}{2n}\left(n-1\right)}{2} = \frac{\left(1-c\right)\left(2n-\left(n-1\right)\right)}{4n} = \frac{\left(1-c\right)\left(n+1\right)}{4n}$$

From (2) we have

$$\pi^i(q_i^*(Q_{-i}^m), Q_{-i}^m) \ = (q_i^*(Q_{-i}^m))^2 = \left(\frac{\left(1-c\right)\left(n+1\right)}{4n}\right)^2 \ = \ \pi_i^m\frac{\pi_i^m}{\pi_i^C}$$

There is therefore a period k, where by deviating for the first time at this period, a firm obtains:

$$\pi_i^m\sum_{t=0}^{k-1}\delta^t + \left(\pi_i^m\frac{\pi_i^m}{\pi_i^C}\right)\delta^k + \pi_i^C\sum_{t=k+1}^{+\infty}\delta^t$$

Any deviation of firm i at period k is unprofitable only if

$$\pi_i^m\sum_{t=0}^{+\infty}\delta^t \geq \pi_i^m\sum_{t=0}^{k-1}\delta^t + \left(\pi_i^m\frac{\pi_i^m}{\pi_i^C}\right)\delta^k + \pi_i^C\sum_{t=k+1}^{+\infty}\delta^t \quad (3)$$

which is equivalent to

$$\sum_{t=k}^{+\infty}\delta^t \geq \frac{\pi_i^m}{\pi_i^C}\delta^k + \frac{\pi_i^C}{\pi_i^m}\sum_{t=k+1}^{+\infty}\delta^t \quad \Longleftrightarrow \quad \left(1-\frac{\pi_i^m}{\pi_i^C}\right)\delta^k + \left(1-\frac{\pi_i^C}{\pi_i^m}\right)\sum_{t=k+1}^{+\infty}\delta^t \geq 0$$

That is, for $\delta\in(0,1)$

$$\left(1-\frac{\pi_i^m}{\pi_i^C}\right) + \left(1-\frac{\pi_i^C}{\pi_i^m}\right)\left(\frac{\delta}{1-\delta}\right) \geq 0 \qquad \Longleftrightarrow \qquad \delta \geq \left(\frac{\pi_i^m}{\pi_i^m+\pi_i^C}\right) = \left(\frac{\left(n+1\right)^2}{\left(n+1\right)^2+4n}\right) := \delta_q(n)$$

We therefore obtain the following result: collusion is sustainable at equilibrium if and only if the firms all have a discount factor greater than $\delta_q(n)$. This Nash equilibrium is also a subgame perfect Nash equilibrium because starting from any subgame we have:

- if no deviation is observed, (3) is satisfied regardless of the current period; and
- if a deviation is observed then the players play a Nash equilibrium of the stage game ($q=\frac{1-c}{n+1}$) forever.

f) (1 pt) From $\frac{\partial\delta_q(n)}{\partial n} = 4\left(\frac{n^2-1}{(n^2+6n+1)^2}\right) > 0$ for $n>1$, we deduce that it is more difficult to maintain a cartel when the number of firms present on the market is high. Because the incentives to deviate are stronger when each member of the cartel receives a smaller share of the overall profit.

g) (2 pts) A similar reasonning can be applied when the aggregate profit $\bar{\pi}$ supported by collusion on an individual quantity $q_i \in [q_i^m = \left(\frac{1-c}{2n}\right), q_i^c = \left(\frac{1-c}{n+1}\right))$. We would obtain another threshold $\delta_q(n,\pi_i^C,\bar{\pi})$ than the previous one denoted as $\delta_q(n,\pi_i^C,\pi^m)$. Indeed, proceeding as in question **(e)** we have

$$\delta_q(n,\pi_i^C,\bar{\pi}) = \left(\frac{\pi_i\left(q_i^*\left(\overline{Q_{-i}}\right),\overline{Q_{-i}}\right)-\frac{\bar{\pi}}{n}}{\pi_i\left(q_i^*\left(\overline{Q_{-i}}\right),\overline{Q_{-i}}\right)-\frac{\pi^C}{n}}\right)$$

where $\overline{Q_{-i}}$ denote firm i's competitors' agregate output sustaining the collusive per-period profit $\overline{\pi}$. From (2), $\pi_i\left(q_i^*\left(Q_{-i}\right),Q_{-i}\right) = \left(\frac{1-c-Q_{-i}}{2}\right)^2 = q_i^*\left(Q_{-i}\right)^2$, so

$$\delta_q(n,\pi_i^C,\overline{\pi}) = \left(\frac{q_i^*\left(\overline{Q_{-i}}\right)^2 - \frac{\overline{\pi}}{n}}{q_i^*\left(\overline{Q_{-i}}\right)^2 - \frac{\pi^C}{n}}\right)$$

We can compare this threshold with $\delta_q(n,\pi_i^C,\pi^m)$. Whether the threshold $\delta_q(n,\pi_i^C,\overline{\pi})$ is increasing or decreasing in $\overline{\pi}$ is unclear:

- On the one hand, the threshold $\delta_q(n,\pi_i^C,\overline{\pi})$ is clearly decreasing in the term $\frac{\overline{\pi}}{n}$. The interpretation is that, for a fixed deviation payoff $q_i^*\left(\overline{Q_{-i}}\right)^2$, the higher the per-period collusive profit $\frac{\overline{\pi}}{n}$, the lower the incentive to deviate.
- On the other hand, from $\pi^C < \overline{\pi}$, this threshold is increasing in $q_i^*\left(\overline{Q_{-i}}\right)^2$. The interpretation is that, for a fixed per-period collusive profit $\frac{\overline{\pi}}{n}$, the higher the deviation payoff $q_i^*\left(\overline{Q_{-i}}\right)^2$, the higher the incentive to deviate. Since $q_i^*\left(.\right)$ is decreasing in Q_{-i} (as from (1) $q_i^*\left(Q_{-i}\right) = \frac{1-c-Q_{-i}}{2}$) the term $q_i^*\left(\overline{Q_{-i}}\right)^2 = \pi_i\left(q_i^*\left(\overline{Q_{-i}}\right),\overline{Q_{-i}}\right)$ is decreasing in $\overline{Q_{-i}}$, and then increasing in $\overline{\pi}$.

The overall incidence of $\bar{\pi}$ on $\delta_q(n,\pi_i^C,\overline{\pi})$ depends on which of these two effects is stronger.