Arbitrage and Pricing – Solution to the Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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April 2026. Duration : 1h30. No document allowed. Calculator allowed.
Answers can be formulated in English or French.

Exercise 1 European Down-and-out and Down-and-in Calls (12 pts)

a) (1 pt) The binomial tree that depicts the evolution of the stock price through time $t \in \{0, 1, 2, 3\}$, is represented by Figure 1.

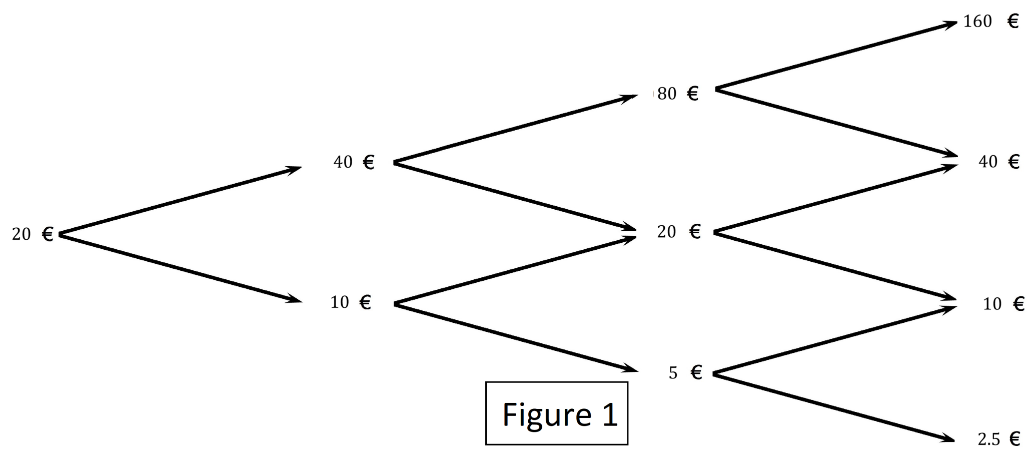

Figure 1

b) (1 pt) No, the binomial tree that depicts the evolution of the derivative price is non-recombining because the barrier of the option requires to taking into account the historical path of realized stock prices.

c) (1 pt) The risk neutral probability is q satisfies $q = \frac{1+r-d}{u-d} = \frac{1.25-0.5}{2-0.5} = 0.5$.

d) (3 pts) The binomial tree that depicts the evolution of the derivative no-arbitrage price, denoted as Do_t, through time $t \in \{0, 1, 2, 3\}$, is represented by Figure 2.

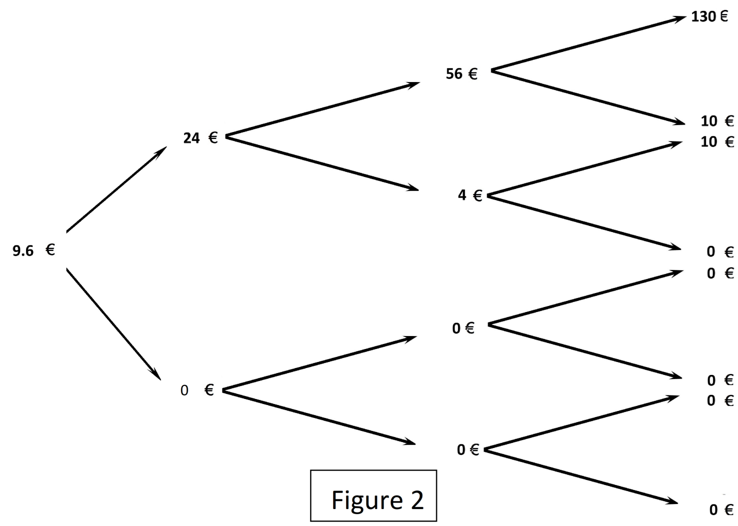

Figure 2

From $S_1^d < B$ we have $Do_t^{d\cdot} = 0$ for every $t \geq 1$. From $S_2^{ud} > B$ we have $Do_3^{u\cdot} = max\,\{0; S_3^{u\cdot} - K\} = (S_3^{u\cdot} - K)^+$. Hence, $Do_3^{uuu} = (S_3^{uuu} - K)^+ = (160 - 30)^+ = 130\,$; $Do_3^{uud} = (S_3^{uud} - K)^+ = (40 - 30)^+ = 10\,$; $Do_3^{udu} = (S_3^{udu} - K)^+ = (40 - 30)^+ = 10\,$; and $Do_3^{udd} = (S_3^{udd} - K)^+ = (7.5 - 30)^+ = 0$.

The remaining option prices, that are Do_2^{uu}, Do_2^{ud}, Do_1^{u}, and Do_0 are obtained by backward induction $Do_2^{uu} = \frac{qDo_3^{uuu}+(1-q)Do_3^{uud}}{1+r} = \frac{70}{1.25} = 56\,$; $Do_2^{ud} = \frac{qDo_3^{udu}+(1-q)Do_3^{udd}}{1+r} = \frac{5}{1.25} = 4\,$; $Do_1^{u} = \frac{qDo_2^{uu}+(1-q)Do_2^{ud}}{1+r} = \frac{30}{1.25} = 24\,$; and $Do_0 = \frac{qDo_1^{u}+(1-q)Do_1^{d}}{1+r} = \frac{12}{1.25} = 9.6$

e) (3 pts) The binomial tree that depicts the evolution of the derivative no-arbitrage price, denoted as Di_t, through time $t \in \{0, 1, 2, 3\}$, is represented by Figure 3.

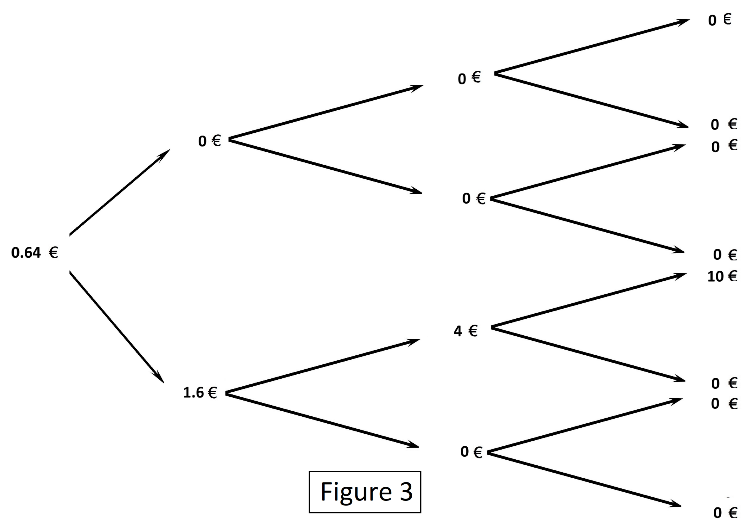

Figure 3

From $S_2^{ud} > B$ and $S_3^{udd} < K$ we have $Di_t^{u\cdot} = 0$ for every $t \geq 1$. From $S_1^d < B$ we have $Di_3^u = max\left\{0; S_3^d - K\right\} = (S_3^d - K)^+$. Hence, $Di_3^{duu} = (S_3^{duu} - K)^+ = (40 - 30)^+ = 10\,$; $Di_3^{dud} = (S_3^{dud} - K)^+ = (10 - 30)^+ = 0\,$; $Di_3^{ddu} = (S_3^{ddu} - K)^+ = (10 - 30)^+ = 0\,$; and $Di_3^{ddd} = (S_3^{ddd} - K)^+ = (2.55 - 30)^+ = 0$.

The remaining option prices, that are $Di_2^{du}, Di_2^{dd}, Di_1^{d}$, and Di_0 are obtained by backward induction

$Di_2^{du} = \frac{qDi_3^{duu}+(1-q)Di_3^{dud}}{1+r}) = \frac{5}{1.25} = 4\,;\ Di_2^{dd} = \frac{qDi_3^{ddu}+(1-q)Di_3^{ddd}}{1+r} = \frac{12}{1.25} = 0\,;\ Di_1^{d} = \frac{qDi_2^{du}+(1-q)Di_2^{dd}}{1+r} = \frac{2}{1.25} = 1.6\,$; and $Di_0 = \frac{qDi_1^{u}+(1-q)Di_1^{d}}{1+r} = \frac{0.8}{1.25} = 0.64$.

f) (2 pts) The binomial tree that depicts the evolution of the derivative no-arbitrage price, denoted as c_t, through time $t \in \{0,1,2,3\}$, is represented by Figure 4.

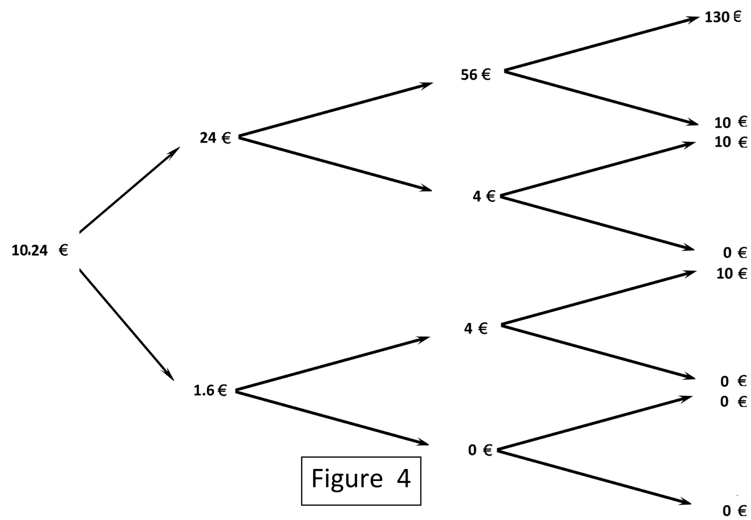

Figure 4

Using that $S_3^{udu} = {S_3}^{duu} = {S_3}^{uud}$ and $S_3^{dud} = S_3^{ddu} = S_3^{udd}$, the derivative no-arbitrage prices at date $t = 3$ write as : $c_3^{uuu} = (S_3^{uuu} - K)^+ = (160-30)^+ = 130\,;\ c_3^{udu} = c_3^{duu} = c_3^{uud} = (S_3^{uud} - K)^+ = (40-30)^+ = 10\,;\ c_3^{dud} = c_3^{ddu} = c_3^{udd} = (S_3^{udd} - K)^+ = (10-30)^+ = 0\,;\ c_3^{ddd} = (S_3^{ddd} - K)^+ = (2.5-30)^+ = 0$.

The derivative no-arbitrage prices at date $t = 2$ write as : $c_2^{uu} = \frac{qc_3^{uuu}+(1-q)c_3^{uud}}{1+r} = \frac{70}{1.25} = 56\,;\ c_2^{ud} = c_2^{du} = \frac{qc_3^{udu}+(1-q)c_3^{udd}}{1+r} = \frac{5}{1.25} = 4\,;\ c_2^{dd} = \frac{qc_3^{ddu}+(1-q)c_3^{ddd}}{1+r} = 0$.

The derivative no-arbitrage prices at date $t = 1$ write as : $c_1^{u} = \frac{qc_2^{uu}+(1-q)c_2^{ud}}{1+r} = \frac{30}{1.25} = 24\,;\ c_1^{d} = \frac{qc_2^{du}+(1-q)c_2^{dd}}{1+r} = \frac{2}{1.25} = 1.6$.

The derivative no-arbitrage price at date t=0 writes as : $c_0 = \frac{qc_1^{u}+(1-q)c_1^{d}}{1+r} = \frac{12.8}{1.25} = 10.24$.

g) (1 pt) Clearly, from d), e), and f), we see that if we combine one European Down-and-out call and one European Down-and-in call, we get the price of the corresponding vanilla European call. Namely, for every time $t \in \{0,1,2,3\}$ and market path $\omega \in \mathcal{P}(\{u,d\}^3)$ we have $c_t^{\omega} = Do_t^{\omega} + Di_t^{\omega}$.

Simultaneously holding the Down-and-out call and the Down-and-in call guarantees that exactly one of the two will pay off identically to the vanilla European call while the other will be worthless. Such a relationship is named "in-out parity".

Exercise 2 Greeks put-call parity (4 pts)

(a) (1 pt)

For a non-dividend paying stock, put--call parity gives at a general time t

$$p_t = c_t - S_t + Ke^{-r(T-t)}.$$

Differentiating with respect to S :

$$\Delta_p \equiv \frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1 = \Delta_c - 1.$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.

(b) (1 pt)

Differentiating with respect to S again :

$$\Gamma_p \equiv \frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2} \equiv \Gamma_c.$$

Hence the gamma of a European put equals the gamma of a European call.

(c) (1 pt)

Differentiating the put--call parity relationship with respect to σ

$$\nu_p \equiv \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma} \equiv \nu_c$$

showing that the vega of a European put equals the vega of a European call.

(d) (1 pt)

Differentiating the put--call parity relationship with respect to t

$$\Theta_p \equiv \frac{\partial p}{\partial t} = \frac{\partial c}{\partial t} + rKe^{-r(T-t)} = \Theta_c + rKe^{-r(T-t)}.$$

Exercise 3 Black-Scholes Portfolio Insurance Delta (4 pts)

a) (2 pts) We can regard the position of all portfolio insurers taken together as a single put option. The known parameters of the option, before the 23% decline, are $S_0 = 70$, $K = (1 - 0.05)\,70 = 66.5$, $T = 1$, $r = 0.03$ and $\sigma = 0.25$. From

$$d_1 = \frac{\ln\left(\frac{70}{66.5}\right) + \left(0.03 + \frac{0.25^2}{2}\right) \times 1}{0.25\sqrt{1}} = 0.4502$$

the delta of the option is

$$\Delta_{\text{put}} = N(d_1) - 1 = N(0.4502) - 1$$

This means that the put behaves like $\Delta_{\text{put}} = N(0.4502) - 1$ shares per 1€ of stock. To hedge the put, administrators must short $-\Delta_{\text{put}}\%$ of the portfolio, that is $70 \times (1 - N(d_1)) = 70 \times (1 - N(0.4502)) = 22.84$€ billion of assets should have been sold before the decline.

Because portfolio insurance replicates a put option with negative delta, a market decline makes the delta more negative, forcing administrators to sell stock to maintain the hedge.

b) (2 pts) After the decline, $S_0 = 70 \times (1 - 0.23) = 53.9$, $K = 66.5$, $T = 1$, $r = 0.03$ and

$\sigma = 0.25$. From

$$d_1 = \frac{\ln\left(\frac{53.9}{66.5}\right) + \left(0.03 + \frac{0.25^2}{2}\right) \times 1}{0.25\sqrt{1}} = -0.5953$$

the delta of the option has dropped to

$$\Delta_{\text{put}} = N(d_1) - 1 = N(-0.5953) - 1$$

This shows that cumulatively $-\Delta_{\text{put}}\%$ of the assets originally held should be sold. The amount of stock that portfolio insurers attempt to sell is therefore

$$-\Delta_{\text{put}} \times S = 53.9 \times (1 - N(d_1)) = 53.9 \times (1 - N(-0.5953)) = 53.9 \times 0.724 \approx 39 \text{ bn €}.$$

After the 23% fall, portfolio insurers attempt to sell about 39 billion euros of stock.