Arbitrage&Pricing

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Chapter 3

Chapter 3: Binomial tree with n period
Outline

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Outline

Introduction

- The one-period model is often too simple for practical purpose.
  - An individual investor has approximately 50 years of adult life when he is making choices over savings, investment and consumption.
    - If important investment decisions are taken every five years, we need at least a 10-period model.
  - Professional investors trade even more frequently.
    - A trader on a stock exchange may adjust his portfolio several times a day resulting in more than 500 investment decisions a month.

Introduction

- The aim of this chapter is to introduce the techniques to asset pricing in a dynamic framework.
  - We use a simple set-up with the European call option as a focus asset in a discrete-time model:
    - to illustrate the backward recursive pricing procedure; and
    - to recover the option price as an unconditional expectation under risk-neutral probabilities.

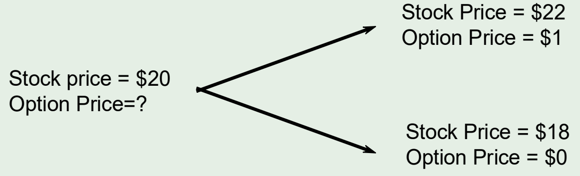
Chapter 3: Binomial tree with n period
Outline

Binomial Trees: Two-Step

- Consider Example D of Chapter 2:

Example

A 3-month call option on the stock has a strike price of 21.



Binomial Trees: Two-Step

- Applying the formula

$$C_0 = \frac{1}{R}\left(\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right)$$

to the 3 months risk-free interest rate of 3.05%, we found the initial price of the option:

$$C_0 = \frac{1}{1.0305}\left(\frac{1.0305-0.9}{1.1-0.9}\right) \simeq 0.633$$

Binomial Trees: Two-Step

- We also obtained the same initial price of the option using a risk-neutral valuation.
- Indeed, by denoting q the probability that gives a return on the stock equal to the risk-free rate:

$$S_0\,(1+r) = S_1^u q + S_1^d (1-q).$$

- The value of the option is

$$C_0 = \frac{C_1^u q + C_1^d (1-q)}{1+r}$$

- So that in Example $D^{(5)}$ we obtained

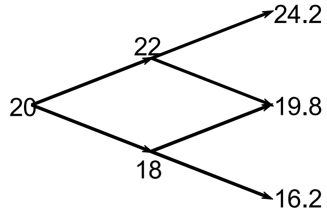
$$20\,(1.0305) = 22q + 18(1-q).$$

so that $q = 0.6525$. And

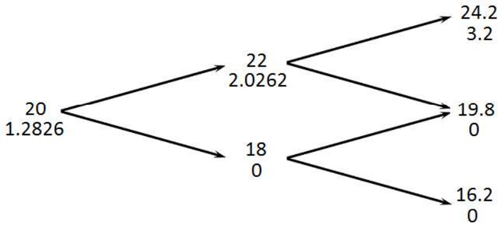
$$C_0 = \frac{1 \times 0.6525 + 0(1-0.6525)}{1.0305} \simeq 0.6332.$$

Binomial Trees: Two-Step

- Let us extend this example to a two-step binomial tree.
- Assume the stock price starts at \$20 and in each of two time steps may go up by 10% or down by 10%.
  - Each time step is 3 months long and 3 months risk-free interest rate of 3.05%.
  - We consider a 6-month option with a strike price of \$21.



Binomial Trees: Two-Step



- When the stock price is 22, the option price is

$$\frac{0.6525 \times 3.2 + 0.3475 \times 0}{1.0305} = 2.0262$$

- When the stock price is 18, the option price is zero, because it leads to two nodes where the option price is zero.
- The initial option price is

$$\frac{0.6525 \times 2.0262 + 0.3475 \times 0}{1.0305} = 1.283$$

Binomial Trees: Two-Step
Generalization

- Suppose that the risk-free interest rate is r, with continuous compounding, and the length of the time step is Δt years.
- We have

$$C_0 = [qC_1^u + (1-q)C_1^d]e^{-r\Delta t}$$

$$q = \frac{e^{r\Delta t} - d}{u-d}$$

$$C_1^u = [qC_2^{uu} + (1-q)C_2^{ud}]e^{-r\Delta t}$$

and

$$C_1^d = [qC_2^{du} + (1-q)C_2^{dd}]e^{-r\Delta t}$$

- So, when $C_2^{du} = C_2^{ud}$ we obtain

$$C_0 = [q^2 C_2^{uu} + 2q(1-q)C_2^{ud} + (1-q)^2 C_2^{dd}]e^{-2r\Delta t}$$

Binomial Trees: Two-Step
A Put Option Example

Question

Consider a 2-year European put with a strike price of \$52 on a stock whose current price is \$50.

We suppose that there are two time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%.

We also suppose that the risk-free interest rate is 5%.

What is the initial price of the option?

Binomial Trees: Two-Step
A Put Option Example

Solution

Binomial Trees: Two-Step
A Put Option Example

Solution

Binomial Trees: Two-Step
A Put Option Example

Solution

Binomial Trees: Two-Step
Delta

Definition

Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.

- It is the number of units of the stock we should hold for each option shorted in order to create a riskless portfolio.
  - It is the same as the Δ introduced earlier in this and previous chapters.
- The construction of a riskless portfolio is sometimes referred to as **delta hedging**.
- The delta of a call option is positive, whereas the delta of a put option is negative.
- The value of Δ varies from node to node.
  - E.g., when the stock price changes from \$18 to \$22, and the option price changes from \$0 to \$1, we have $\Delta = \frac{1-0}{22-18} = 0.25$.

Chapter 3: Binomial tree with n period
Outline

Binomial tree: generalization
Basic notions on Probability

Definition

A **filtration** is a sequence of $\sigma-$algebra $(\mathcal{F}_k)_{1\leq k\leq n}$, such that each $\sigma-$algebra in the sequence contains all the sets contained by the previous $\sigma-$algebra. Formally, $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \forall k < n$.

- A filtration models the evolution of information through time. So for example, if it is known by time k whether or not an event, E, has occurred, then we have $E \in \mathcal{F}_k$.

Definition

Let $\mathcal{F} := (\mathcal{F}_k)_{1\leq k\leq n}$ be a filtration. The stochastic process $(X_k)_{1\leq k\leq n}$ is $\mathcal{F}-$**adapted**, if X_k is $\mathcal{F}_k-$measurable for each $k \leq n$.

- The idea is that the value of X_k is known at time k when the information represented by $\mathcal{F}_k$ is known.

Binomial tree: generalization
Basic notions on Probability

Proposition (3.1)

If the stochastic process $(X_k)_{1\leq k\leq n}$ is $\mathcal{F}-$adapted, then X_i is $\mathcal{F}_k-$measurable for any $i \leq k$.

Proof.

Straightforward.

If the stochastic process $(X_k)_{1\leq k\leq n}$ is $\mathcal{F}-$adapted then X_i is $\mathcal{F}_i-$measurable.

Since $\mathcal{F} :=(\mathcal{F}_k)_{1\leq k\leq n}$ is a filtration then $\mathcal{F}_i \subseteq \mathcal{F}_k, \forall i \leq k$, with $k \leq n$.

So, X_i is $\mathcal{F}_k-$measurable. □

Binomial tree: generalization
Basic notions on Probability

Definition

The **natural filtration** of the stochastic process $(X_k)_{1\leq k\leq n}$ is given by the smallest filtration $\mathcal{F}$ for which $(X_k)_{1\leq k\leq n}$ is $\mathcal{F}-$adapted.

We denote it by $\mathcal{F}^X := (\mathcal{F}_k^X)_{1\leq k\leq n}$, with $\mathcal{F}_k^X$ the $\sigma-$algebra generated by X_k.

I.e., $\mathcal{F}_k^X := \sigma(X_1, X_2, ..., X_k)$.

Binomial tree: generalization
Basic notions on Probability

Definition (new)

A stochastic process M :=$(M_k)_{1\leq k\leq n}$ is a $\mathcal{F}-$**martingale** under $\mathbb{P}$ if M is $\mathcal{F}-$adapted,

$$\mathbb{E}^{\mathbb{P}}[|M_k|] < +\infty, \text{ for any } k \leq n$$

and

$$\mathbb{E}^{\mathbb{P}}[M_{k+1}|\mathcal{F}_k] = M_k.$$

- If the previous equality is replaced with $\leq$ the process tends to go down and is called a *supermartingale*. If the previous equality is replaced with $\geq$ the process tends to go up and is called a *submartingale*.
- So, a supermartingale (resp. submartingale) is a loosing (resp. winning) game.

Binomial tree: generalization
Basic notions on Probability

- If M :=$(M_k)_{1\leq k\leq n}$ is a $\mathcal{F}-$**martingale** under $\mathbb{P}$ then

$$\mathbb{E}^{\mathbb{P}}[M_k|\mathcal{F}_i] = M_i \text{ , for any } i \leq k$$

and in particular, we have

$$\mathbb{E}^{\mathbb{P}}[M_k] = M_0.$$

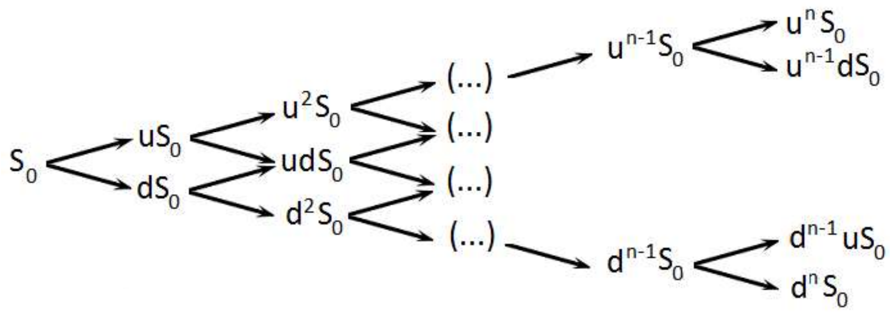
Binomial tree: generalization
Set-up

- We extend the model of the previous chapter to n periods.
- We consider an interval of time $[0, T]$ divided into n periods: $0 = t_0 \leq t_1 \leq ... \leq t_n = T$.
- There are two assets:
  - A *non-risky asset* S_t^0:

$$1 \rightarrow (1+r) \rightarrow (1+r)^2 \rightarrow ... \rightarrow (1+r)^n$$

  - A *risky asset* S_t that evolves according to the following Table

Binomial tree: generalization
Set-up



- The order of occurrence of u and d's does not count. So, the tree recombines (e.g., $du^2S_0 = udu\ S_0 = u^2dS_0$). At time t the asset may then only take $t+1$ values. (If the order would have count we would have obtained 2^t values.)

Binomial tree: generalization
Set-up

- At date t Nature selects $\omega_t \in \{\omega_t^u, \omega_t^d\}$. So

  $\Omega := \{(\omega_1, \omega_2, ..., \omega_n)\,|\forall i \in \{0, 1, ..., n\}$ we have $\omega_i = \omega_i^u$ or $\omega_i = \omega_i^d\}$.

  We assume that the probability of occurrence of u is time-invariant:

$$\mathbb{P}(\omega_i = \omega_i^u) = p$$

  and

$$\mathbb{P}\left(\omega_i = \omega_i^d\right) = 1 - p.$$

Binomial tree: generalization
Set-up

- So

$$\mathbb{P}(\omega_1, \omega_2, ..., \omega_n) = p^{\#\{i \in \{0,1,...,n\}|\omega_i = \omega_i^u\}} \times (1-p)^{\#\{i \in \{0,1,...,n\}|\omega_i = \omega_i^d\}}.$$

- The value of the asset at time t_i, can be written as

$$S_{t_i} = S_0 \prod_{k=0}^{i} Y_k$$

  with $(Y_k)_{k=0,...,n} : \Omega^{n+1} \longmapsto \{u, d\}^{n+1}$ being a collection of random variables i.i.d., with Y_k is realized at time k, and takes the value u with probability p and d with probability $(1-p)$.

Binomial tree: generalization
Set-up

- So we have

$$\mathbb{P}(Y_i = u) = \mathbb{P}(\omega_i = \omega_i^u) = p$$

  and

$$\mathbb{P}(Y_i = d) = \mathbb{P}\left(\omega_i = \omega_i^d\right) = 1 - p.$$

Binomial tree: generalization
Set-up

- The information available at time t_i is given by the filtration $(\mathcal{F}_{t_k})_{1\leq k\leq i}$, with

$$\mathcal{F}_{t_i} := \sigma(\omega_1, \omega_2, ..., \omega_i) = \sigma(Y_1, Y_2, ..., Y_i) = \sigma(S_{t_1}, S_{t_2}, ..., S_{t_i}).$$

Definition (Mathematics)

In our market, a **derivative** is a random variable that is $\mathcal{F}_T-$measurable.

- So, a derivative takes the form of a function $\phi(S_{t_1}, S_{t_2}, ..., S_{t_i})$.

Chapter 3: Binomial tree with n period
Outline

Binomial tree: generalization
Simple portfolio strategies

Definition

A **simple portfolio strategy** consists in an initial amount of cash x and a stochastic process $\Delta := (\Delta_k)_{0\leq k\leq n-1}$ which is $\mathcal{F}-$adapted.

We denote this strategy by the pair (x, Δ) and its value at date t_i by $X_{t_i}^{x,\Delta}$.

- A simple portfolio strategy consists in using a part of an initial amount of cash x to buy (at the initial date) the risky asset in quantity Δ_0, and to invest the other part of x in a non-risky asset.
  - Then at date t_i we invest into the risky asset in quantity Δ_i.
  - The process is $\mathcal{F}-$adapted because the amount of investment at date t_i is determined using the information available at date t_i.
  - This simple portfolio strategy is self-financing.

Binomial tree: generalization
Simple portfolio strategies

- Between period t_i and t_{i+1}, the portfolio takes the form of Δ_i units of risky asset and $\frac{X_{t_i}^{x,\Delta}-\Delta_i S_{t_i}}{(1+r)^i}$ units of non-risky asset. So, the value of the portfolio at time t_i is given by

$$X_{t_i}^{x,\Delta} = \Delta_i S_{t_i} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i}(1+r)^i.$$

- Since the strategy is self-financing, no money is withdrawn nor inserted during the time interval $[t_i, t_{i+1})$, so we have

$$X_{\mathbf{t}_{i+1}}^{x,\Delta} = \Delta_i S_{\mathbf{t}_{i+1}} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i}(1+r)^{\mathbf{i+1}}.$$

- Let $\tilde{Z}$ denotes the **current value** of the variable Z at date $t = 0, 1, ..., n$.
  - So, the current value of the portfolio X at date t_i writes as

$$\tilde{X}_{t_i}^{x,\Delta} := \frac{X_{t_i}^{x,\Delta}}{(1+r)^i}$$

  - and the current value of the risky asset at date t_i writes as

$$\tilde{S}_{t_i} := \frac{S_{t_i}}{(1+r)^i}.$$

- So we have

$$\begin{aligned}
\tilde{X}_{t_i+1}^{x,\Delta} &= \frac{X_{t_i+1}^{x,\Delta}}{(1+r)^{i+1}} \\
&= \frac{\Delta_i S_{t_{i+1}} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i}(1+r)^{i+1}}{(1+r)^{i+1}} \\
&= \Delta_i \tilde{S}_{t_i+1} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i} \\
&= \Delta_i \tilde{S}_{t_i+1} + \left(\tilde{X}_{t_i}^{x,\Delta} - \Delta_i \tilde{S}_{t_i}\right).
\end{aligned}$$

- From

$$\begin{aligned}
\tilde{X}_{t_i}^{x,\Delta} &= \tilde{X}_{t_i}^{x,\Delta} + \left(\Delta_i \tilde{S}_{t_i} - \Delta_i \tilde{S}_{t_i}\right) \\
&= \Delta_i \tilde{S}_{t_i} + \left(\tilde{X}_{t_i}^{x,\Delta} - \Delta_i \tilde{S}_{t_i}\right)
\end{aligned}$$

we obtain the self-financing condition

$$\tilde{X}_{t_{i+1}}^{x,\Delta} - \tilde{X}_{t_i}^{x,\Delta} = \Delta_i \left(\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i}\right). \quad (1)$$

- This condition can be rewritten as

$$\tilde{X}_{t_{i+1}}^{x,\Delta} = x + \sum_{k=0}^{i} \Delta_k \left(\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k}\right).$$

Definition

A **simple arbitrage** is a simple portfolio strategy that gives to a portfolio no value at time $t = 0$ and a value at time $T = t_n$ which is strictly positive with positive probability and is never negative. Formally, it is a pair $(x = 0, \Delta)$ with $\Delta \in \mathbb{R}^n$ such that

$$X_T^{0,\Delta} \geq 0 \text{ and } \mathbb{P}(X_T^{0,\Delta} > 0) > 0.$$

Definition

We say that there is **no simple arbitrage opportunity** (**NAO'**) if

$$\forall \Delta \in \mathbb{R}^n \; \{X_T^{0,\Delta} \geq 0 \implies X_T^{0,\Delta} = 0 \; \mathbb{P} - a.s.\}$$

Binomial tree: generalization
Arbitrage and risk-neutral probability

Proposition (3.2)

If NAO' then $d < 1 + r < u$.

Proof.

We proceed by contradiction. Assume NAO' and $d \geq 1 + r$.

Consider the following simple arbitrage strategy:
- buy one unit of the risky asset; and

- sell the equivalent amount of the non risky asset in period $t = 0$;

- then resell the unit of the risky asset at time t_1; and

- invest it into the non-risky asset until period T

- (i.e., $x = 0$, $\Delta_0 = 1$, and $\Delta_i = 0$ for any $i \geq 1$). (...) □

Binomial tree: generalization
Arbitrage and risk-neutral probability

Proof.

Such a strategy is deterministic so it is $\mathcal{F}-$adapted. At date $T = t_n$ the portfolio value is given by:

$$\begin{aligned} \tilde{X}_T^{0,\Delta} &= 0 + \sum_{k=0}^{n-1} \Delta_k \left(\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k} \right) \\ &= \tilde{S}_{t_1} - \tilde{S}_{t_0} \end{aligned}$$

(...) □

Binomial tree: generalization
Arbitrage and risk-neutral probability

Proof.

Since S_{t_1} can only takes two values (either uS_0 or dS_0), the portfolio value at time T is either

$$(1+r)^n S_0 \left(\frac{u}{(1+r)} - 1 \right) > 0$$

or

$$(1+r)^n S_0 \left(\frac{d}{(1+r)} - 1 \right) \geq 0$$

which contradicts NAO', since both values occur with strictly positive probabilities (resp. p and $(1-p)$). (...) □

Binomial tree: generalization
Arbitrage and risk-neutral probability

Proof.

Similarly, we obtain a contradiction by assuming $u \leq 1 + r$ and by considering the simple arbitrage strategy that consists in:

- selling one unit of the risky asset; and

- buying the equivalent amount of the non risky asset in period $t = 0$;

- I.e., $x = 0$, $\Delta_0 = -1$, and $\Delta_i = 0$ for any $i \geq 1$. □

- Consider the following probability on Ω:

$$\mathbb{Q}(\omega_1, \omega_2, ..., \omega_n) = q^{\#\{i \in \{1,...,n\} | \omega_i = \omega_i^u\}} \times (1-q)^{\#\{i \in \{1,...,n\} | \omega_i = \omega_i^d\}}$$

with

$$q := \frac{(1+r)-d}{u-d}.$$

- We then have

$$\mathbb{Q}(S_{t_i} = uS_{t_{i-1}}) = \mathbb{Q}(Y_i = u) = q$$

and

$$\mathbb{Q}(S_{t_i} = dS_{t_{i-1}}) = \mathbb{Q}(Y_i = d) = 1-q.$$

- Let us show that $\mathbb{Q}$ is a risk-neutral probability measure.

- By definition (see chap. 2), a risk-neutral probability measure or equivalent martingale measure (EMM) is a probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ and for which any simple strategy expressed in current value is a martingale.

Proposition (3.3)

$\tilde{S} := \left(\tilde{S}_{t_i}\right)_{i \in \{1,2,...,n\}}$ *is a $\mathcal{F}$−martingale under $\mathbb{Q}$.*

Proof.

Clearly, $\tilde{S}_{t_i} := \frac{S_{t_i}}{(1+r)^i}$ is $\mathcal{F}_{t_i}$−measurable for each $i \leq n$, so $\tilde{S}$ is $\mathcal{F}$−adapted.(...) □

Proof.

Moreover, we have

$$\mathbb{E}^{\mathbb{Q}}[|\tilde{S}_{t_i}|] = \frac{\mathbb{E}^{\mathbb{Q}}[|S_{t_i}|]}{(1+r)^i} < +\infty, \text{ for any } i \leq n$$

and (...) □

Proof.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t_{i+1}} | \mathcal{F}_{t_i}] &= \frac{qu\tilde{S}_{t_i} + (1-q)\,d\tilde{S}_{t_i}}{1+r} \\
&= \frac{1}{1+r}\left(\frac{(1+r)-d}{u-d}u\tilde{S}_{t_i} + \left(1 - \frac{(1+r)-d}{u-d}\right)d\tilde{S}_{t_i}\right) \\
&= \frac{1}{1+r}\left(\frac{(1+r)-d}{u-d}u + \frac{u-(1+r)}{u-d}d\right)\tilde{S}_{t_i} \\
&= \frac{1}{1+r}\left(\frac{(1+r)(u-d)}{u-d}\right)\tilde{S}_{t_i} = \tilde{S}_{t_i}.
\end{aligned}$$

□

Binomial tree: generalization
Arbitrage and risk-neutral probability

- The following result states that if the current values of the standard assets are martingale under a given probability then it is so of the current value of any simple portfolio strategy.

Proposition (3.4)

The current value $\tilde{X}^{x,\Delta}$ of any simple portfolio strategy (x,Δ) is a $\mathcal{F}-$martingale under $\mathbb{Q}$.

Proof.

Clearly, $\tilde{X}^{x,\Delta}$ is $\mathcal{F}-$adapted. Moreover, we have

$$\mathbb{E}^{\mathbb{Q}}[|\tilde{X}^{x,\Delta}_{t_i}|] < +\infty, \text{ for any } i \leq n.$$

(...) □

Binomial tree: generalization
Arbitrage and risk-neutral probability

Proof.

Now it suffices to show that

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}^{x,\Delta}_{t_{i+1}} - \tilde{X}^{x,\Delta}_{t_i}|\mathcal{F}_{t_i}] = 0.$$

From (1) we have

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}^{x,\Delta}_{t_{i+1}} - \tilde{X}^{x,\Delta}_{t_i}|\mathcal{F}_{t_i}] = \mathbb{E}^{\mathbb{Q}}[\Delta_i \left(\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i}\right)|\mathcal{F}_{t_i}]$$

so

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}^{x,\Delta}_{t_{i+1}} - \tilde{X}^{x,\Delta}_{t_i}|\mathcal{F}_{t_i}] = \Delta_i \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i}|\mathcal{F}_{t_i}] = 0$$

where the last equality comes from the previous Proposition. □

Binomial tree: generalization
Arbitrage and risk-neutral probability

Theorem (3.5)

If $d < R < u$ then there is an equivalent martingale measure $\mathbb{Q}$.

Proof.

According to the previous result we know that $\mathbb{Q}$ is a probability measure for which any simple strategy expressed in current value is a martingale. Moreover, $\mathbb{Q}$ is equivalent to $\mathbb{P}$ since $d < R < u$ implies that $q \in (0,1)$ and that $\mathbb{Q}(\omega_1, \omega_2, ..., \omega_n) > 0$ for every $(\omega_1, \omega_2, ..., \omega_n) \in \Omega$. □

The value at date t_i of any simple portfolio strategy writes as

$$X^{x,\Delta}_{t_i} = \frac{\mathbb{E}^{\mathbb{Q}}[X^{x,\Delta}_T|\mathcal{F}_{t_i}]}{(1+r)^{n-i}}.$$

Binomial tree: generalization
Arbitrage and risk-neutral probability

- Hence, if we are able to hedge a derivative, NAO implies that the value of the hedging portfolio at date t_i is given by the expected current value of its final value under the risk-neutral probability.
- Before exploiting this idea, let us state the following result.

Proposition (3.6)

If there is an equivalent martingale measure $\mathbb{Q}$ then NAO' holds.

Proof.

As in the previous chapter, let $\Delta \in \mathbb{R}^n$ such that $X_T^{0,\Delta} \geq 0$.

Since $\mathbb{Q}$ is an equivalent martingale measure, we have

$$\mathbb{E}^{\mathbb{Q}}\left[X_T^{x=0,\Delta}\right] = x = 0.$$

Which means that $X_T^{0,\Delta}$ is a random variable that is positive and whose expected value is zero.

This variable is then equal to zero $\mathbb{Q} - a.s.$

Finally, since $\mathbb{Q}$ is equivalent to $\mathbb{P}$ we obtain $\mathbb{P}\left(X_T^{0,\Delta} > 0\right) = 0$. □

- From Proposition 3.2

$$NAO' \Longrightarrow d < R < u$$

- From Theorem 3.5

$$d < R < u \Longrightarrow \text{ there is an equivalent martingale measure}$$

- From Proposition 3.6 we have

$$\text{there is an equivalent martingale measure } \Longrightarrow NAO'$$

- Hence we obtain

$$\begin{aligned} NAO' &\Longleftrightarrow d < R < u \\ &\Longleftrightarrow \text{ there is an equivalent martingale measure.} \end{aligned}$$

- Saying that

  "the current values of every standard asset is martingale under Q"

  is then equivalent to say that

  "the current value of every simple portfolio strategy is martingale under Q".

Chapter 3: Binomial tree with n period
Outline

Binomial tree: generalization
Hedging derivative

Theorem (3.7)

In our market, every derivative is replicable by using a simple portfolio strategy (x, Δ).

Question

What is the form of (x, Δ)?

- We are looking at for a simple portfolio strategy (x, Δ) replicating a derivative of value C_T at date T. Since C_T is $\mathcal{F}_{t_n}-$adapted, the value of the derivative takes the form of a function $\phi(S_{t_1}, S_{t_2}, ..., S_{t_n})$ so (x, Δ) has to satisfy

$$X_{t_n}^{x,\Delta} = \phi(S_{t_1}, S_{t_2}, ..., S_{t_n}).$$

Binomial tree: generalization
Hedging derivative

- According to Proposition 3.4, the current value of every simple portfolio strategy is martingale under the EMM $\mathbb{Q}$, so the value $X_{t_k}^{x,\Delta}$ of the replicating portfolio at date t_k satisfies

$$X_{t_k}^{x,\Delta} = \frac{\mathbb{E}^{\mathbb{Q}}\left[X_{t_n}^{x,\Delta} | \mathcal{F}_{t_k}\right]}{(1+r)^{n-k}} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi(S_{t_1}, S_{t_2}, ..., S_{t_n}) | \mathcal{F}_{t_k}\right]}{(1+r)^{n-k}}.$$

- So the initial amount of our replicating portfolio has to be

$$x := \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi(S_{t_1}, S_{t_2}, ..., S_{t_n})\right]}{(1+r)^{n}}. \quad (2)$$

Binomial tree: generalization
Hedging derivative

- Now, since $\frac{1}{(1+r)^{n-k}}\mathbb{E}^{\mathbb{Q}}\left[\phi(S_{t_1}, S_{t_2}, ..., S_{t_n}) | \mathcal{F}_{t_k}\right]$ is a random variable which is $\mathcal{F}_{t_k}-$measurable it can be rewritten as a function $V_k(S_{t_1}, S_{t_2}, ..., S_{t_k})$ where $V_k(\cdot)$ is deterministic. Let

$$V_k(S_{t_1}, S_{t_2}, ..., S_{t_k}) := \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi(S_{t_1}, S_{t_2}, ..., S_{t_n}) | \mathcal{F}_{t_k}\right]}{(1+r)^{n-k}}. \quad (3)$$

- In the previous chapter which introduces the model with one period, we have seen that the quantity of the risky asset Δ of the replicating portfolio looks like the variation of the value of the derivative induced by the variation of the underlying asset.

Hedging derivative

- In the proof of Theorem, we shall take $\Delta := (\Delta)_{k\in\{1,2,...,n\}}$ satisfying that for any $k \in \{1, 2, ..., n\}$

$$\Delta_k := \frac{V_{k+1}(S_{t_1}, S_{t_2}, ..., S_{t_k}, uS_{t_k}) - V_{k+1}(S_{t_1}, S_{t_2}, ..., S_{t_k}, dS_{t_k})}{u\,S_{t_k} - dS_{t_k}}. \quad (4)$$

- Observe that for any $k \in \{1, 2, ..., n\}$, Δ_k is $\mathcal{F}_{t_k}-$measurable as a function of $(S_{t_1}, S_{t_2}, ..., S_{t_k})$. So, Δ is $\mathcal{F}-$adapted and (x, Δ) is indeed a simple portfolio strategy.
- Now, let us establish the proof of Theorem 3.7 according to which the simple portfolio strategy (x, Δ) with Δ satisfying (4) replicates our derivative.

Proof.

We have to show that

$$X_{t_n}^{x,\Delta} = \phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right) = V_n\left(S_{t_1}, ..., S_{t_n}\right).$$

Let us proceed by induction. Let $P(k)$, $k \in \{1, 2, ..., n\}$ be the following statement:

$$X_{t_k}^{x,\Delta} = V_k\left(S_{t_1}, ..., S_{t_k}\right)$$

Clearly, $P(0)$ is true. Indeed, we have $X_{t_0}^{x,\Delta} = x$ and by (2)

$$x := \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)\right]}{(1+r)^n} = \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)|\mathcal{F}_{t_0}\right]}{(1+r)^{n-0}}$$

which by (3) correspond to $V_0\left(S_{t_0}\right)$. (...) □

Proof.

Assume $P(k)$ is true. Let us show that $P(k+1)$ is true.
$P(k)$ writes as

$$\begin{aligned} X_{t_k}^{x,\Delta} &= V_k\left(S_{t_1}, ..., S_{t_k}\right) \\ &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)|\mathcal{F}_{t_k}\right]}{(1+r)^{n-k}} \\ &= \frac{1}{(1+r)} \frac{\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)|\mathcal{F}_{t_k}\right]}{(1+r)^{n-k-1}} \\ &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)}{(1+r)^{n-(k+1)}}\right| \mathcal{F}_{t_k}\right]. \end{aligned}$$

(...) □

Proof.

Now using that for any martingale Z that is $\mathcal{F}-$measurable we have for any k and s such that $k + s < n$

$$\mathbb{E}\left[Z_{t_n}|\mathcal{F}_{t_k}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_{t_n}|\mathcal{F}_{t_{k+s}}\right]|\mathcal{F}_{t_k}\right]$$

we obtain

$$\begin{aligned} X_{t_k}^{x,\Delta} &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}}\left[\left.\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\phi\left(S_{t_1}, S_{t_2}, ..., S_{t_n}\right)}{(1+r)^{n-(k+1)}}\right| \mathcal{F}_{t_{k+1}}\right]\right| \mathcal{F}_{t_k}\right] \\ &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}}\left[V_{k+1}\left(S_{t_1}, ..., S_{t_{k+1}}\right)\right| \mathcal{F}_{t_k}\right]. \end{aligned}$$

(...) □

Proof.

So,

$$\begin{aligned} X_{t_k}^{x,\Delta} &= \frac{\mathbb{E}^{\mathbb{Q}}\left[\left.\begin{array}{c} V_{k+1}\left(S_{t_1}, ..., S_{t_k}, u\, S_{t_k}\right) \mathbf{1}_{\left\{Y_{t_{k+1}}=u\right\}} \\ +V_{k+1}\left(S_{t_1}, ..., S_{t_k}, d\, S_{t_k}\right) \mathbf{1}_{\left\{Y_{t_{k+1}}=d\right\}} \end{array}\right| \mathcal{F}_{t_k}\right]}{(1+r)} \\ &= \frac{\begin{array}{c} \mathbb{Q}\left(Y_{t_{k+1}} = u\right) V_{k+1}\left(S_{t_1}, ..., S_{t_k}, u\, S_{t_k}\right) \\ +\mathbb{Q}\left(Y_{t_{k+1}} = d\right) V_{k+1}\left(S_{t_1}, ..., S_{t_k}, d\, S_{t_k}\right) \end{array}}{(1+r)} \end{aligned}$$

□

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Proof.

That is

$$X_{t_k}^{x,\Delta} = \frac{\left[qV_{k+1}\left(S_{t_1},...,S_{t_k},u\,S_{t_k}\right)+(1-q)\,V_{k+1}\left(S_{t_1},...,S_{t_k},d\,S_{t_k}\right)\right]}{(1+r)} \tag{5}$$

Now, from (1) we have

$$\tilde{X}_{t_{k+1}}^{x,\Delta} = \tilde{X}_{t_k}^{x,\Delta} + \Delta_k\left(\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k}\right)$$

so

$$\frac{X_{t_{k+1}}^{x,\Delta}}{(1+r)^{k+1}} = \frac{X_{t_k}^{x,\Delta}}{(1+r)^k} + \Delta_k\left(\frac{S_{t_{k+1}}}{(1+r)^{k+1}} - \frac{S_{t_k}}{(1+r)^k}\right)$$

(...) □

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Proof.

which rewrites as

$$X_{t_{k+1}}^{x,\Delta} = X_{t_k}^{x,\Delta}(1+r) + \Delta_k\left(S_{t_{k+1}} - (1+r)\,S_{t_k}\right)$$

Using (5) and (4) we obtain

$$\begin{aligned} X_{t_{k+1}}^{x,\Delta} &= qV_{k+1}\left(S_{t_1},...,S_{t_k},u\,S_{t_k}\right)+(1-q)\,V_{k+1}\left(S_{t_1},...,S_{t_k},d\,S_{t_k}\right) \\ &+ \frac{V_{k+1}\left(S_{t_1},S_{t_2},...,S_{t_k},uS_{t_k}\right) - V_{k+1}\left(S_{t_1},S_{t_2},...,S_{t_k},dS_{t_k}\right)}{u\,S_{t_k} - dS_{t_k}} \\ &\times\left(S_{t_{k+1}} - (1+r)\,S_{t_k}\right). \end{aligned}$$

(...) □

Binomial tree: generalization
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Proof.

By replacing $S_{t_{k+1}}$ with $Y_{k+1}S_{t_k}$ and q with $\frac{1+r-d}{u-d}$ we have

$$\begin{aligned} X_{t_{k+1}}^{x,\Delta} &= V_{k+1}\left(S_{t_1},...,S_{t_k},u\,S_{t_k}\right)\frac{Y_{k+1}-d}{u-d} \\ &+V_{k+1}\left(S_{t_1},...,S_{t_k},d\,S_{t_k}\right)\frac{u-Y_{k+1}}{u-d}. \end{aligned}$$

Since Y_{k+1} can only takes the value u and d we obtain

$$X_{t_{k+1}}^{x,\Delta} = V_{k+1}\left(S_{t_1},...,S_{t_k},Y_{k+1}\,S_{t_k}\right) = V_{k+1}\left(S_{t_1},...,S_{t_k},S_{t_{k+1}}\right)$$

which is $P(k+1)$. (...) □

Binomial tree: generalization
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Proof.

Since every derivative is replicable, under NAO, a derivative of final value

$$C_T = \phi\left(S_{t_1},S_{t_2},...,S_{t_n}\right)$$

has a value at date t_k given by

$$C_{t_k} = \frac{1}{(1+r)^{n-k}}\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1},S_{t_2},...,S_{t_n}\right)|\mathcal{F}_{t_k}\right]$$

and in particular at date 0

$$C_0 = \frac{1}{(1+r)^n}\mathbb{E}^{\mathbb{Q}}\left[\phi\left(S_{t_1},S_{t_2},...,S_{t_n}\right)\right].$$

□

Binomial tree: generalization
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- It means that the derivative price at any date can be obtained by backward induction.
  - we can treat each binomial step separately and work back from the end of the life of the option to the beginning to obtain the current value of the option.
- The following result extends Proposition 2.6 of Chapter 2 to our setup.

Proposition (3.8)

If every asset is replicable with a simple portfolio strategy (complete market) then the equivalent martingale measure is unique.

Proof.

The proof is the one of Proposition 2.6. □

Chapter 3: Binomial tree with n period
Outline

Conclusion

- The binomial model with n periods produces similar results to the model with one period:
  - the derivative price does not depend the probabilities of up, p, and down, $(1-p)$, movements in the stock price at each node of the tree.
  - the derivative price is the expected current value, expressed with the equivalent martingale measure $\mathbb{Q}$, of its future value.
  - the quantity Δ of the risky asset in the replicative portfolio measures how the derivative price moves with the underlying asset price.
- When stock price movements are governed by a multistep binomial tree, we can use backward induction to deduce the initial option price from the final option price.

Conclusion

- We can assume the world is risk-neutral when valuing an option.
  - No-arbitrage arguments and risk-neutral valuation are equivalent and lead to the same option prices.
- The delta of a stock option, Δ, considers the effect of a small change in the underlying stock price on the change in the option price.
  - It is the ratio of the change in the option price to the change in the stock price.
  - For a riskless position, an investor should buy Δ shares for each option sold.
  - An inspection of a typical binomial tree shows that delta changes during the life of an option.
  - This means that to hedge a particular option position, we must change our holding in the underlying stock periodically.