Arbitrage&Pricing

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Chapter 2

Chapter 2: Binomial tree with one period
Outline

Introduction
Motivation

- A useful and very popular technique for pricing an option involves constructing a **binomial tree**.
  - This is a diagram representing different possible paths that might be followed by the stock price over the life of an option.
  - The underlying assumption is that the stock price follows a *random walk*.

Chapter 2: Binomial tree with one period
Outline

Basic notions on Probability (Part 1)

- A useful and very popular technique for pricing an option involves constructing a binomial tree.
- Let $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a *probability space*. That is, a triple of:
  - Ω a *sample space* which is the universe of possible outcomes;
  - $\mathcal{F}$ a *set of events*, where an event is a subset of Ω;
  - $\mathbb{P}$ a *probability function* from $\mathcal{F}$ to $[0,1]$, which measures the likeliness that an event will occur.

Basic notions on Probability (Part 1)

Example (Flipping a coin)

$\Omega = \{H, T\}$, $\mathcal{F} = \{\varnothing, \{H\}, \{T\}, \Omega\}$,

$$\mathbb{P}: \quad \mathcal{F} \longmapsto [0,1]$$
$$A \longmapsto \mathbb{P}[A] = \begin{cases} 1 \text{ if } A = \Omega \\ \frac{1}{2} \text{ if } A = \{H\} \text{ or } \{T\} \\ 0 \text{ if } A = \varnothing \end{cases} .$$

Basic notions on Probability (Part 1)

- Observe that for the probability $\mathbb{P}$ to be well-defined, the set of events $\mathcal{F}$ has to satisfy some properties:
  - $\mathcal{F}$ is non-empty;
  - $\mathcal{F}$ is closed under complementation: If A is in $\mathcal{F}$, then so is its complement, $\Omega \backslash A$; and
  - $\mathcal{F}$ is closed under countable unions: If A_1, A_2, A_3, ... are in $\mathcal{F}$, then so is $A = A_1 \cup A_2 \cup A_3 \cup$ ... .
- We say that such $\mathcal{F}$ is a $\sigma-$algebra (or $\sigma-$field).
  - In general, we will take $\mathcal{F}$ as the smallest $\sigma-$algebra generated by the experiment.

Basic notions on Probability (Part 1)

Example (A)

We consider the experiment that consists in rolling a dice and then checking whether the number 6 is the outcome.

So, $\Omega = \{1,2,3,4,5,6\}$ and $\mathcal{F}_A = \{\varnothing, \{6\}, \{1,2,3,4,5\}, \Omega\}$.

Example (B)

We consider the experiment that consists in rolling a dice and then checking whether the outcome is even.

So, $\Omega = \{1,2,3,4,5,6\}$ and $\mathcal{F}_B = \{\varnothing, \{2,4,6\}, \{1,3,5\}, \Omega\}$.

Basic notions on Probability (Part 1)

Example (C)

We consider the experiment that consists in rolling a dice and then checking the outcome.

So, $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{F}_C = 2^\Omega$, where 2^Ω denotes the *power set* of the sample space.

I.e., $\mathcal{F}_C$ has $2^6 = 64$ elements. E.g., one of this element is $\{2, 5\}$, which consists in checking whether the outcome is 2 or 5.

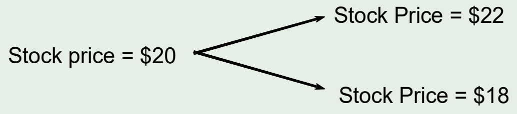
Chapter 2: Binomial tree with one period
Outline

Binomial tree (Part 1)

- We consider a market with only two periods: $t = 0$ and $t = 1$.
- There are two assets.
  - A riskless asset who values 1 at date $t = 0$ and $R = (1 + r)$ at date $t = 1$. r denotes the risk-free interest rate that we could obtain with a zero coupon.
  - A risky asset S who values S_0 at date $t = 0$ and can take two different values at date $t = 1$: $S_1 \in \{S_1^u, S_1^d\}$ with $S_1^u = uS_0$, $S_1^d = dS_0$, and $d < u$.

Example (D)

A stock price is currently \$20 and in 3 months it will be either \$22 or \$18



Binomial tree (Part 1)

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the probability space corresponding to this market situation. We have:
  - $\Omega = \{\omega_u, \omega_d\}$;
  - $\mathcal{F}_0 = \{\varnothing, \Omega\}$; $\mathcal{F}_1 = \{\varnothing, \{\omega_u\}, \{\omega_d\}, \Omega\}$; and
  - $\mathbb{P}$ such that $\mathbb{P}(\omega_u) = p$ and $\mathbb{P}(\omega_d) = 1 - p$, with $p \in (0, 1)$.
- Observe that $\mathcal{F}_0 \subset \mathcal{F}_1$: we acquire information through time.

Chapter 2: Binomial tree with one period
Outline

Basic notions on Probability (Part 2)

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a *probability space*. A (real-valued) **random variable** is a (real) function $X : \Omega \longmapsto \mathbb{R}$ such that $\{\omega \in \Omega | X(\omega) \leq x\} \in \mathcal{F}$ for every $x \in \mathbb{R}$.

- Said differently, a random variable is a function that assigns a numerical value to each state of the world, $X : \Omega \longmapsto \mathbb{R}$, such that the values taken by X are known to someone who has access to the information $\mathcal{F}$.

Basic notions on Probability (Part 2)

Example (A')

The window will be opened if and only if the maximal number of the dice is realized.

By associating the number 1 to the action of opening the window and zero otherwise, we have:

$$X_A = \begin{cases} 1 \text{ if } \omega = \{6\} \\ 0 \text{ otherwise.} \end{cases}$$

Observe that we can use $\mathcal{F}_A$ (or $\mathcal{F}_C$) but not $\mathcal{F}_B$.

Basic notions on Probability (Part 2)

Example (B')

You earn 100€ if the realization of the dice is even and you lose 50€ otherwise.

$$X_B = \begin{cases} +100 \text{ if } \omega \in \{2, 4, 6\} \\ -50 \text{ otherwise.} \end{cases}$$

Observe that we can use $\mathcal{F}_B$ (or $\mathcal{F}_C$) but not $\mathcal{F}_A$.

Example (C')

You earn 15€ if the realization of the dice is 5 and zero otherwise.

$$X_C = \begin{cases} +15 \text{ if } \omega = \{5\} \\ 0 \text{ otherwise.} \end{cases}$$

Observe that we can use $\mathcal{F}_C$ but neither $\mathcal{F}_A$ nor $\mathcal{F}_B$.

Definition

Let $\mathcal{F}$ denotes a $\sigma-$algebra associated with Ω.

A (real) function $X : \Omega \longmapsto \mathbb{R}$ is $\mathcal{F}-$**measurable** if, for any two numbers $a, b \in \mathbb{R}$, all the states of the world $\omega \in \Omega$ for which X takes value between a and b forms a set that is an event (an element of $\mathcal{F}$).

Formally, $\forall a, b \in \mathbb{R}$, $a < b$, we have $\{\omega \in \Omega | a < X(\omega) < b\} \in \mathcal{F}$.

- So, a random variable is $\mathcal{F}-$measurable if and only if it is known with the information given by $\mathcal{F}$.
  - I.e., for any two numbers, we are able to answer the question on whether the realization of the random variable belongs to the interval formed by these two numbers.
  - Roughly speaking, we are able to say what actually happened.

Example (A")

X_A is $\mathcal{F}_A-$measurable and $\mathcal{F}_C-$measurable but is not $\mathcal{F}_B-$measurable.

Example (B")

X_B is $\mathcal{F}_B-$measurable and $\mathcal{F}_C-$measurable but is not $\mathcal{F}_A-$measurable.

Example (C")

X_C is $\mathcal{F}_C-$measurable but neither $\mathcal{F}_A-$measurable nor $\mathcal{F}_B-$measurable.

- A more general definition is that a function $X : G \longmapsto H$ is **measurable** if the preimage under X of every element in the $\sigma-$algebra associated with H is in the $\sigma-$algebra associated with G.
  - Formally, if $\mathcal{G}$ (resp. $\mathcal{H}$) is the $\sigma-$algebra associated to G (resp. H), then $X^{-1}(y) := \{g \in G | X(g) = y\} \in \mathcal{G}, \forall y \in \mathcal{H}$.
  - The idea is that a measurable function pulls back measurable sets.
- The notion of measurability depends on the $\sigma-$algebras that are used.
  - In our definition, as the $\sigma-$algebra associated with $\mathbb{R}$ we took the Borel $\sigma-$algebra on the reals, i.e., the smallest $\sigma-$algebra on $\mathbb{R}$ which contains all the intervals.

Chapter 2: Binomial tree with one period
Outline

Binomial tree (Part 2)

- The risky asset S (who values S_0 at date $t = 0$ and can take two different values at date $t = 1$: $S_1 \in \{S_1^u, S_1^d\}$) is a random variable that is $\mathcal{F}_1-$measurable, but is not $\mathcal{F}_0-$measurable.
- That is, the information known at date 0 is not sufficient to say what is the realization of S.
  - Instead, we have to wait until date 1.
- Observe that $\mathcal{F}_1$ is the smallest $\sigma-$algebra that makes S measurable.

Binomial tree (Part 2)

Definition (Finance)

A **derivative** is a contract that derives its value from the performance of an underlying entity (e.g., asset, index, interest rate, ...)

Definition (Mathematics)

In our market, a **derivative** is a random variable that is $\mathcal{F}_1-$measurable.

- The value of the derivative depends on the realization of the underlying variables at date $t = 1$.
- If S_1 is the underlying asset, then any derivative can be written as a $\mathcal{F}_1-$measurable function of S_1.

Example

A call with underlying x and strike K is a derivative that takes the form $\phi : x \longmapsto (x - K)^+$.

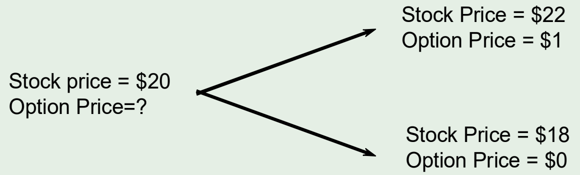
Binomial tree (Part 2)

Question

What is the value of a derivative at date $t = 0$?

Example (D')

A 3-month call option on the stock has a strike price of 21.



Binomial tree (Part 2)

- To tackle the question we will use a portfolio that replicates the derivative.
- Namely, we will build two self-financing portfolios.
  - One such a portfolio uses the risky asset while the other does not.
  - Both portfolios are built so that they take the same value at date $t = 1$.
  - NAO then implies that they have same value at date $t = 0$.

Binomial tree (Part 2)
Simple portfolio strategies

Definition

A **simple portfolio strategy** consists in using a part of an initial amount of cash x to buy (at the initial date) a risky asset in quantity Δ, and to invest the other part of x in a non-risky asset.

We denote this strategy by the pair (x, Δ) and its value at date t by $X_t^{x,\Delta}$.

- By definition, in our setup, we have
$$X_0^{x,\Delta} = \Delta S_0 + (x - \Delta S_0)\, 1 = x. \quad (1)$$
and
$$X_1^{x,\Delta} = \Delta S_1 + (x - \Delta S_0)\, R = xR + \Delta\,(S_1 - S_0 R)\,.$$
- This strategy is self-financing. It is called *simple* because it only uses standard assets: the non-risky and the risky ones.

Binomial tree (Part 2)
Simple portfolio strategies

Theorem (2.1)

In our market, every derivative is replicable by using a simple portfolio strategy (x, Δ).

Proof.

□

Binomial tree (Part 2)
Simple portfolio strategies

Proof.

□

Binomial tree (Part 2)
Simple portfolio strategies

- So, under NAO, the price of a derivative in period $t = 0$ is given by
$$\begin{aligned} C_0 &= X_0^{x,\Delta} = x \\ &= \frac{1}{R}\left(\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right) \end{aligned}$$
which is a weighted sum of its future values C_1^u and C_1^d.

Example (D”)

Assume the 3 months risk-free rate is 3.05% .
We then obtain
$$C_0 = \frac{1}{1.0305}\left(\frac{1.0305 - 0.9}{1.1 - 0.9}\right) \simeq 0.633$$

Binomial tree (Part 2)
Simple portfolio strategies

- A market where every asset is replicable with a simple portfolio strategy is said to be **complete**.
- Now let us study how the initial value of a simple portfolio strategy, $X_0^{x,\Delta}$, depends on its future value, $X_1^{x,\Delta}$.

Definition

A **simple arbitrage** is a simple portfolio strategy that gives to a portfolio no value at time $t = 0$ and a value at time $t = 1$ which is strictly positive with positive probability and is never negative.

Formally, it is a pair $(x = 0, \Delta)$ with $\Delta \in \mathbb{R}$ such that

$$X_1^{0,\Delta} \geq 0 \text{ and } \mathbb{P}(X_1^{0,\Delta} > 0) > 0.$$

Binomial tree (Part 2)
Simple portfolio strategies

Definition

We say that there is **no simple arbitrage opportunity** (**NAO'**) if

$$\forall \Delta \in \mathbb{R}, \ \{X_1^{0,\Delta} \geq 0 \implies X_1^{0,\Delta} = 0 \ \mathbb{P} - a.s.\}$$

Proposition (2.2)

If NAO' then $d < R < u$.

Binomial tree (Part 2)
Simple portfolio strategies

Proof.

□

Binomial tree (Part 2)
Simple portfolio strategies

Proof.

□

Binomial tree (Part 2)
Simple portfolio strategies

- Let $\tilde{X}$ denotes the **current value** of the portfolio X at date $t = 0, 1$:

$$\tilde{X}_t^{x,\Delta} := \frac{X_t^{x,\Delta}}{R^t}.$$

  - So, we have
$$\tilde{X}_0^{x,\Delta} = x$$
and
$$\begin{aligned}\tilde{X}_1^{x,\Delta} &= \frac{xR + \Delta(S_1 - S_0R)}{R} \\ &= x + \Delta\left(\frac{S_1}{R} - S_0\right) = x + \Delta(\tilde{S}_1 - S_0)\end{aligned}$$
  - In term of current values, the portfolio self-financing condition then writes as
$$\tilde{X}_1^{x,\Delta} - \tilde{X}_0^{x,\Delta} = \Delta(\tilde{S}_1 - \tilde{S}_0).$$

Chapter 2: Binomial tree with one period
Outline

Basic notions on Probability (Part 3)

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a *probability space*. A **stochastic process** is a collection of random variables on Ω, indexed by a totally ordered set T (e.g., referring to time).

Formally, a stochastic process X is a collection $(X_t)_{t\in T}$ where each X_t is a random variable on Ω.

- When $T = \{1, 2, ..., n\}$ the stochastic process is discrete. We will denote it as $(X_k)_{1\leq k\leq n}$.

Basic notions on Probability (Part 3)

Definition (Preliminary)

A **martingale** is a stochastic process with finite means, in which the conditional expectation of the next value, given the current and preceding values, is the current value.

Formally, the stochastic process $(X_t)_{t\in T}$ is a martingale if for any time n we have
$$\mathbb{E}[|X_n|] < +\infty$$
and
$$\mathbb{E}[X_{n+1}|X_1, ..., X_n] = X_n.$$

Basic notions on Probability (Part 3)

- Originally, martingale referred to a class of betting strategies that was popular in 18th-century France.
- The simplest of these strategies was designed for a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails.
  - The strategy had the gambler double his bet after every loss so that the first win would recover all previous losses plus win a profit equal to the original stake.
  - As the gambler's wealth and available time jointly approach infinity, his probability of eventually flipping heads approaches 1, which makes the martingale betting strategy seem like a sure thing.
  - Observe that the exponential growth of the bets eventually bankrupts its users.

Basic notions on Probability (Part 3)

Definition

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ (on the same sample space Ω) are said to be **equivalent** if they define the same null sets. Formally, for any event $A \in \Omega$, $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$.

Definition (Preliminary)

A **risk-neutral probability measure** or **equivalent martingale measure (EMM)** is a probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ and for which any simple strategy expressed in current value is a martingale. Formally,

$$\tilde{X}_0^{x,\Delta} = \mathbb{E}^{\mathbb{Q}}\left[\tilde{X}_1^{x,\Delta}\right]$$

or equivalently

$$X_0^{x,\Delta} = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}\left[X_1^{x,\Delta}\right].$$

Basic notions on Probability (Part 3)

- The idea is that an "equivalent martingale" measure is a probability measure under which the current value of all financial assets at time t is equal to the expected future payoff of the asset discounted at the risk-free rate, given the information structure available at time t.
  - Equivalently, a "risk-neutral" probability measure is a probability measure under which the underlying risky asset has the same expected return as the non risky asset.

Chapter 2: Binomial tree with one period
Outline

Binomial tree (Part 3)

Proposition (2.3)

If $d < R < u$ then there is an equivalent martingale measure $\mathbb{Q}$.

Proof.

□

Binomial tree (Part 3)

Proof.

□

Binomial tree (Part 3)

Proof.

□

Binomial tree (Part 3)

Proposition (2.4)

If there is an equivalent martingale measure $\mathbb{Q}$ then NAO' holds.

Proof.

□

Binomial tree (Part 3)

- Following the two previous propositions, we have

$$NAO' \Longrightarrow d < R < u \Longrightarrow \text{there is an equivalent martingale measure} \Longrightarrow NAO'.$$

- Hence we obtain

$$NAO' \Longleftrightarrow d < R < u \Longleftrightarrow \text{ there is an equivalent martingale measure.}$$

Binomial tree (Part 3)
Evaluating and hedging derivative

Proposition (2.5)

Assume NAO. The price of a derivative at time $t = 0$ is given by

$$C_0 = \frac{\mathbb{E}^{\mathbb{Q}}[C_1]}{1+r} = \frac{1}{R}\left(\mathbb{Q}(\omega_u)\, C_1^u + \mathbb{Q}(\omega_d)\, C_1^d\right) = \frac{1}{R}\left(qC_1^u + (1-q)\, C_1^d\right)$$

Proof.

Exercise. (Hint: Straightforwardly obtained from the previous section) □

Binomial tree (Part 3)
Evaluating and hedging derivative

- Observe that the equivalent martingale measure does not depend on the probabilities p (and $1-p$) of the state ω_u (and ω_d).
  - So, the price of an option is independent from the probability behind the evolution of the underlying asset.
    - This is partly due to the fact that the replicating portfolio contains the underlying asset.
  - To determine the price of the derivative we then just need to know r, u, and d.

Question

How to determine u and d?

- We shall see how this is correlated with the volatility of the asset.

Binomial tree (Part 3)
Evaluating and hedging derivative

- In the replicating portfolio, the quantity of the risky asset is given by

$$\Delta = \frac{C_1^u - C_1^d}{(u-d)\, S_0} = \frac{\phi(S_1^u) - \phi(S_1^d)}{(u-d)\, S_0}.$$

  - This quantity measures how the price of the option varies with the underlying asset price variation.

Example ($D^{(4)}$)

We then have

$$\Delta = \frac{1-0}{(1.1-0.9)\, 20} = 0.25$$

Binomial tree (Part 3)
Evaluating and hedging derivative

Example ($D^{(4)}$)

The hedging strategy consists then:

- in buying 0.25 unit of the risky asset (the cost is $\Delta S_0 = 0.25 \times 20 = 5$); and

- to invest $(x - \Delta S_0) = \frac{1}{1.0305}\frac{1.0305-0.9}{1.1-0.9} - 5 \simeq -4.3668$ into the non-risky asset.

Doing so, we indeed obtain

$$-4.366\,8 \times 1.0305 + 0.25 \times 22 = 1 = C_1^u$$

and

$$-4.366\,8 \times 1.0305 + 0.25 \times 18 \simeq 0 = C_1^d.$$

Binomial tree (Part 3)
Evaluating and hedging derivative

- An alternative way to determine Δ is to consider a portfolio consisting of a long position in Δ shares of the risky asset and a short position in one call option, and then to calculate the value of Δ that makes this portfolio riskless.
  - If there is an up movement in the stock price, the value of the portfolio at the end of the life of the option is

$$S_0 u\Delta - C_1^u$$

  - If there is a down movement in the stock price, the value becomes

$$S_0 d\Delta - C_1^d$$

  - The two are equal (i.e., $S_0 u\Delta - C_1^u = S_0 d\Delta - C_1^d$) when

$$\Delta = \frac{C_1^u - C_1^d}{(u-d)\,S_0}.$$

Binomial tree (Part 3)
Evaluating and hedging derivative

- The value of the portfolio:
  - at time 1 is $S_0 u\Delta - C_1^u$ ($= S_0 d\Delta - C_1^d$);
  - today is $\frac{S_0 u\Delta - C_1^u}{1+r}$;
- Another expression for the portfolio value today is $S_0\Delta - C_0$
- Hence

$$C_0 = S_0\Delta - \frac{S_0 u\Delta - C_1^u}{1+r}$$

- Substituting for $\Delta = \frac{C_1^u - C_1^d}{S_0 u - S_0 d}$ we obtain

$$C_0 = \frac{qC_1^u + (1-q)C_1^d}{1+r}$$

where $q = \frac{1+r-d}{u-d}$, which confirms Proposition 2.5.

Binomial tree (Part 3)
Evaluating and hedging derivative

Proposition (2.6)

If every asset is replicable with a simple portfolio strategy (complete market) then the equivalent martingale measure is unique.

Proof.

□

Proof.

Thus, we have

$$\frac{\mathbb{E}^{\mathbb{Q}_1}[\mathbf{1}_{\mathcal{B}}]}{R} = x = \frac{\mathbb{E}^{\mathbb{Q}_2}[\mathbf{1}_{\mathcal{B}}]}{R}$$

Moreover, $\mathbf{1}_{\mathcal{B}}$ denoting the indicator function, for any probability $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\mathcal{B}}] = \mathbb{Q}(\mathcal{B})$$

So we obtain

$$\mathbb{Q}_1(\mathcal{B}) = \mathbb{Q}_2(\mathcal{B})$$

That is, $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are the same. □

- It is natural to interpret q and $1-q$ as probabilities of up and down movements.
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate.
- When the probability of an up and down movements are q and $1-q$ the expected stock price at time 1 is $S_0(1+r)$.
- This shows that the stock price earns the risk-free rate.

- Binomial trees illustrate the general result that to value a derivative we can assume that:
  - The expected return on a stock (or any other investment) is the risk-free rate.
  - The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.
- This is known as using risk-neutral valuation.

- q is the probability that gives a return on the stock equal to the risk-free rate:
$$S_0(1+r) = S_1^u q + S_1^d(1-q).$$
- The value of the option is
$$C_0 = \frac{C_1^u q + C_1^d(1-q)}{1+r}$$

Binomial tree (Part 3)
Evaluating and hedging derivative

Example ($D^{(5)}$)

We have

$$20\,(1.0305) = 22q + 18(1-q).$$

so that $q = 0.6525$. And

$$C_0 = \frac{1 \times 0.6525 + 0(1-0.6525)}{1.0305} \simeq 0.6332.$$

Binomial tree (Part 3)
Evaluating and hedging derivative

Question

What is the Call price of a Call with S_0 =100, K =100, r =0.05, d =0.9 and u =1.1?

Give a hedging strategy and depict a tree that illustrates the replication.

Solution

Binomial tree (Part 3)
Evaluating and hedging derivative

Solution

Binomial tree (Part 3)
Evaluating and hedging derivative

Solution

Binomial tree (Part 3)
Evaluating and hedging derivative

Question

What about a Put with the same characteristics?

Solution

Binomial tree (Part 3)
Evaluating and hedging derivative

Solution

Binomial tree (Part 3)
Evaluating and hedging derivative

Question

Does the Call-Put parity holds?

Solution