Arbitrage&Pricing

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LEDa

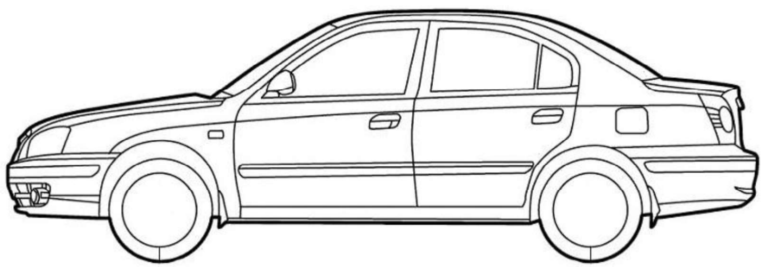
Chapter 1

Chapter 1: Introduction to Arbitrage opportunities

Outline

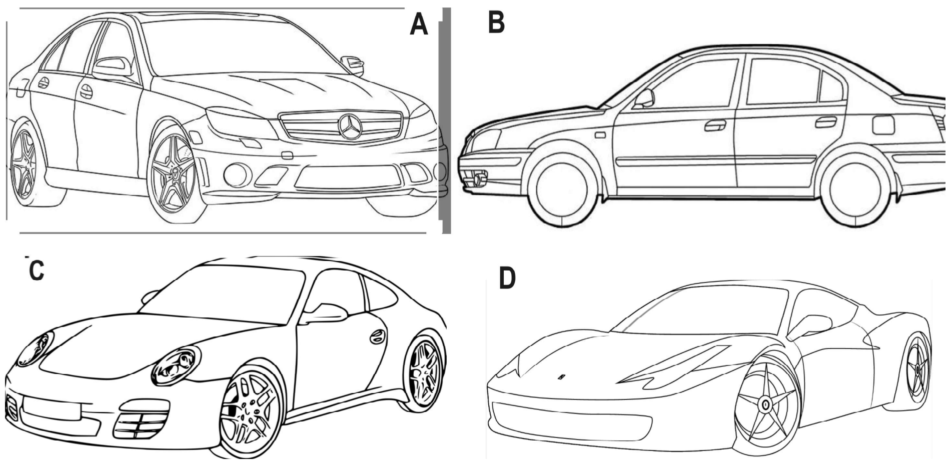
Introduction

How much is this car worth?



Introduction

Rank these cars in order of price



Introduction
What is the shortest way from Paris to NYC?

Introduction
What is the shortest way from Paris to NYC?

Chapter 1: Introduction to Arbitrage opportunities
Outline

Setup

Throughout the following we will assume:

1. All assets are infinitely divisible;
2. The market is liquid: we can sell and buy at anytime;
3. We can short sell and borrow;
4. There is no transaction cost; and
5. We can borrow and lend at the same interest rate, r.

Chapter 1: Introduction to Arbitrage opportunities
Outline

Arbitrage

- What are the possible market scenarios?
  - Let us denote:
    - Ω: *Sample space*, i.e., the set of all possible market outcomes;
    - $\mathbb{P}$: *Probability* measure over all possible market outcomes.
- What are the possible investment strategy?

Definition

A trading strategy is **self-financing** if all trades are financed by selling or buying assets in the portfolio. No money is withdrawn or inserted after the initial forming of the portfolio.

- So, a self-financing strategy requires any purchase of new assets to be financed by the sale of old ones.

Definition

A **self-financing portfolio** is a portfolio that is managed through a self-financing strategy.

Arbitrage

- Denote by X_t the *portfolio value* in time t.

Definition

An **arbitrage** over the period $[0; T]$ is a self-financing strategy that gives to a portfolio no value at time $t = 0$ and a value X_T at time T which is strictly positive with positive probability and is never negative. Formally, $X_0 = 0$, $\mathbb{P}(X_T > 0) > 0$ and $X_T \geq 0$.

- An arbitrage:
  - is then an investment that has a non-positive cost at $t = 0$ but has a positive probability of yielding a positive payoff at $t = T$ and never produces a negative payoff.
  - satisfies that starting from zero, we never lose and there are real profit opportunities.
  - E.g., to buy a stock that costs nothing, but that will possibly generate dividend income in the future.

Arbitrage

- The existence of “free lunches” can be regarded as a market inefficiency in the sense that certain assets are not priced consistently with each other.
- On financial markets, there exist practitioners called *arbitrageurs* paid to detect arbitrage opportunities.
  - All things being equal, according to the law of supply and demand, the action of these investors will cause the market price of the stock to move in the direction that quickly eliminates the arbitrage.
- In quantitative finance, we assume that arbitrage opportunities do not exist since if they did, market forces would quickly act to dispel them.

Arbitrage

Definition

When there is no arbitrage over the period $[0;T]$, we say that there is **no-arbitrage opportunity** (**NAO**, no free lunch). Formally,

$$\{X_0 = 0 \text{ and } X_T \geq 0\} \Longrightarrow P(X_T > 0) = 0.$$

- The basic idea of what no arbitrage means, is that there is no free lunch – you cannot get money out of nothing.

Chapter 1: Introduction to Arbitrage opportunities
Outline

Portfolios comparison

- Another underlying idea of the NAO assumption is that two portfolios which have the same cost at time 0 and same payoffs in all possible cases, must cost the same thing.
  - More formally, we have the next results.
  - Before stating these results, we need the following definition.

Definition

A **zero-coupon bond** (or zero-coupon), maturing at time T, pays 1 at time T and nothing before time T. Its price at time t, is denoted by $B(t;T)$.

Portfolios comparison

- So, it takes $(T-t)$ periods for a zero-coupon of price $B(t;T)$ to worth 1.

$$B(t;T) \xrightarrow{T-t} 1$$

  - Or, equivalently, after $(T-t)$ periods, one unit worth $\frac{1}{B(t;T)}$.

$$1 \xrightarrow{T-t} \frac{1}{B(t;T)}.$$

- Observe that we have $B(t;T) \leq 1$, so $\frac{1}{B(t;T)} \geq 1$.

Portfolios comparison

- In the following, we will assume that if we deposit X_t in cash to a bank account, then at time T we will get the amount $\frac{X_t}{B(t;T)}$.
- Now, we can state our first result.

Proposition (1.1)

Assume NAO. If two self-financing portfolios X and Y have the same value in time T, then they have the same value in time 0. Formally,

$$X_T = Y_T \Longrightarrow X_0 = Y_0.$$

Portfolios comparison

Proof.

□

Portfolios comparison

Proof.

□

Portfolios comparison

- The arbitrage strategy used in the proof is really simple:
  - It consists in time 0 to buy the portfolio with the lower price (X_0); and
  - to sell the portfolio with the higher price (Y_0).
  - Since both portfolios have the same value in T, we make a positive profit.

Chapter 1: Introduction to Arbitrage opportunities
Outline

Call-Put parity

Definition

A **call** (resp. **put**) **option** gives the holder of the option the right to buy (resp. sell) an asset by a certain date for a certain price.

The date specified in the contract is known as the **expiration date** or the **maturity date**.

The price specified in the contract is known as the **exercise price** or the **strike price**.

Definition

American options can be exercised at any time up to the expiration date.

European options can be exercised only on the expiration date itself.

Call-Put parity

- Let K denotes the strike price, T the maturity, and S_T the final price of the underlying asset.
- The payoff from a long position in a European call option is

$$\max(S_T - K, 0) := (S_T - K)^+$$

  - This reflects the fact that the option will be exercised if $S_T > K$ and will not be exercised if $S_T \leq K$.
  - The payoff to the holder of a short position is

$$-\max(S_T - K, 0) = \min(K - S_T, 0)$$

- The payoff to the holder of a long position in a European put option is

$$\max(K - S_T, 0) := (K - S_T)^+$$

  - and the payoff from a short position is

$$-\max(K - S_T, 0) = \min(S_T - K, 0).$$

Call-Put parity

- Consider a European call and a European put, both with the identical strike price K and same maturity T. Denote the price of the call (resp. put) at date t by C_t (resp. P_t).

Proposition (1.2: Call-Put Parity)

If NAO holds, then

$$C_t - P_t = S_t - KB(t, T).$$

Call-Put parity

Proof.

□

Call-Put parity

Proof.

Chapter 1: Introduction to Arbitrage opportunities
Outline

Pricing a Forward contract

Definition

A **forward contract** is an agreement to buy or sell an asset at a certain future time for a certain price. The price is fixed at the date of the agreement, but is paid at the time of the delivery of the asset.

- Let $F(t,T)$ denote the price of the Forward, agreed on date t for the delivery of the asset S at time T.

Proposition (1.3)

If NAO holds, then $F(t,T) = \frac{S_t}{B(t,T)}$.

Pricing a Forward contract

Proof.

□