Arbitrage and Pricing – Solution to the Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

Jérôme MATHIS (LEDa)

April 2025. Duration : 1h30. No document allowed. Calculator allowed.
Answers can be formulated in English or French.

Exercise 1 (4 pts) **a) (2 pts)** At date 1, the price of the call option is either $C_1^u = \max\{4.5-2.5;0\} = 2$ or $C_1^d = \max\{2-2.5;0\} = 0$.

The equivalent martingale measure is

$$p = \left(\frac{1+r-d}{u-d}\right) = \frac{1.08-\frac{2}{3}}{\frac{4.5}{3}-\frac{2}{3}} = \frac{3.24-2}{2.5} = \frac{1.24}{2.5} \approx 0.496$$

as $r = 8\%$, $u = \frac{S_1^u}{S_0} = \frac{4.5}{3}$, and $d = \frac{S_1^d}{S_0} = \frac{2}{3}$.

The non-arbitrage price of the call option is then

$$C_0 = \frac{pC_1^u + (1-p)C_1^d}{1+r} = \frac{\frac{1.24}{2.5}\times 2 + \frac{1.26}{2.5}\times 0}{1.08} \approx 0.91852 \approx 0.92$$

b) (2 pts) The market price of the option is below its non-arbitrage price. An arbitrage portfolio could then consist in, at time 0 :

- buying the option (spending 0.9€) ;
- short-selling $\Delta_0 = \frac{C_1^u - C_1^d}{S_1^u - S_1^d} = \frac{2-0}{4.5-2} = \frac{2}{2.5} = 0.8$ shares of the stock (receiving $\Delta_0 S_0 = 0.8\times 3 = 2.4$€) ; and
- investing the difference $(2.4 - 0.9 = 1.5)$ in the money market.

At time 1 :

- if there is an upward move, we need $0.8 \times 4.5 = 3.6$€ to buy and deliver 0.8 shares of the stock. We receive $C_1^u = 2$€ from the option, and $1.5 \times 1.08 = 1.62$€ from the money market. So, the granted profit is :
$$2 + 1.62 - 3.6 = 0.02$$
- if there is a downward move, we need $0.8 \times 2 = 1.6$€ to buy and deliver 0.8 shares of the stock. We receive $C_1^d = 0$€ from the option, and $1.5 \times 1.08 = 1.62$€ from the money market. So, the granted profit is :
$$1.62 - 1.6 = 0.02$$

Hence, in both cases the granted profit is 0.02€.

Exercise 2 (7 pts) **a) (3 pts)** Consider a portfolio consisting of (-1) derivative φ_0 and $+\Delta$ shares of derivative D_0. In six months, if derivative D's price goes to D_1^u, the portfolio is worth $D_1^u\Delta - \varphi(D_1^u)$. If derivative D's price goes to D_1^d, it is worth $D_1^d\Delta - \varphi(D_1^d)$. These are the same when

$$D_1^u\Delta - \varphi(D_1^u) = D_1^d\Delta - \varphi(D_1^d)$$

or

$$\Delta = \frac{\varphi(D_1^u) - \varphi(D_1^d)}{D_1^u - D_1^d}.$$

The value of the portfolio in six months is $D_1^u \times \left(\frac{\varphi(D_1^u)-\varphi(D_1^d)}{D_1^u - D_1^d}\right) - \varphi(D_1^u)$ in any case. Its value today must be the present value of

$$D_1^u\Delta - \varphi(D_1^u)$$

or

$$\left(D_1^u\Delta - \varphi(D_1^u)\right) e^{-r/2}.$$

This means that

$$-\varphi_0 + D_0\Delta = \left(D_1^u\Delta - \varphi(D_1^u)\right) e^{-r/2}$$

$\iff$

$$\begin{aligned}
\varphi_0 e^{r/2} &= D_0\Delta e^{r/2} - \left(D_1^u\Delta - \varphi(D_1^u)\right) \\
&= \Delta\left(D_0 e^{r/2} - D_1^u\right) + \varphi(D_1^u) \\
&= \frac{\left(\varphi(D_1^u) - \varphi(D_1^d)\right)\left(D_0 e^{r/2} - D_1^u\right) + \varphi(D_1^u)\left(D_1^u - D_1^d\right)}{D_1^u - D_1^d} \\
&= \varphi(D_1^u)\left(\frac{D_0 e^{r/2} - D_1^d}{D_1^u - D_1^d}\right) + \varphi(D_1^d)\left(\frac{D_1^u - D_0 e^{r/2}}{D_1^u - D_1^d}\right).
\end{aligned}$$

Therefore,

$$\varphi_0 = e^{-r/2}\left(\varphi(D_1^u)\left(\frac{D_0 e^{r/2} - D_1^d}{D_1^u - D_1^d}\right) + \varphi(D_1^d)\left(\frac{D_1^u - D_0 e^{r/2}}{D_1^u - D_1^d}\right)\right).$$

b) (2 pts) We can calculate the probability, q, of an up movement in a risk-neutral world. This must satisfy :

$$D_0 e^{r/2} = qD_1^u + (1-q)D_1^d$$

so that

$$q = \frac{D_0 e^{r/2} - D_1^d}{D_1^u - D_1^d}.$$

The value of the option is then its expected payoff discounted at the risk-free rate :

$$\begin{aligned}
\varphi_0 &= e^{-r/2}\left(\varphi(D_1^u)q + \varphi(D_1^d)(1-q)\right) \\
&= e^{-r/2}\left(\varphi(D_1^u)\left(\frac{D_0 e^{r/2} - D_1^d}{D_1^u - D_1^d}\right) + \varphi(D_1^d)\left(\frac{D_1^u - D_0 e^{r/2}}{D_1^u - D_1^d}\right)\right).
\end{aligned}$$

This agrees with the result of part **a)**.

c) (2 pts)

i) When $r = 0\%$, the value D_0 in a risk-neutral world satisfies $D_0 = qD_1^u + (1-q)D_1^d$ where $D_1^u = (S_1^u)^2 = 36$, $D_1^d = (S_1^d)^2 = 9$, and q denotes the risk-neutral probability.

This probability solves

$$S_0 = qS_1^u + (1-q)S_1^d.$$

That is,

$$q = \frac{S_0 - S_1^d}{S_1^u - S_1^d} = \frac{5-3}{6-3} = \frac{2}{3}.$$

So,

$$D_0 = \frac{2}{3} \times 36 + \frac{1}{3} \times 9 = 27.$$

ii) By definition,

$$\varphi(D_1^u) = \max\{K - D_1^u, 0\} = \max\{25 - 36, 0\} = 0$$

and

$$\varphi(D_1^d) = \max\{K - D_1^d, 0\} = \max\{25 - 9, 0\} = 16.$$

From part (b), we then have

$$\begin{aligned}\varphi_0 &= \varphi(D_1^u)\left(\frac{D_0 - D_1^d}{D_1^u - D_1^d}\right) + \varphi(D_1^d)\left(\frac{D_1^u - D_0}{D_1^u - D_1^d}\right) \\ &= 16 \times \left(\frac{36-27}{36-9}\right) = \frac{16}{3} \approx 5.33.\end{aligned}$$

Exercise 3 (9 pts) **a) (3 pts)** From time T_1, the underlying asset price of this derivative instrument is known. At time $t = T_2$ the value of the derivative is

$$f = \ln S_{T_1}.$$

At time $t = T_2 - 1$ the value of the derivative is

$$f = e^{-r} \ln S_{T_1}.$$

More generally, at any time $t \in [T_1; T_2]$ the value of the derivative is

$$f = e^{-r(T_2 - t)} \ln S_{T_1}.$$

Before time T_1, the value of the derivative has to take into account the random path followed by the returns. More precisely, from time $t < T_1$, taking into account that volatility erodes returns, the log return of the stock price at time T_1 satisfies

$$\ln R_{T_1 - t} \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T_1 - t), \sigma^2(T_1 - t)\right).$$

The stock price at time T_1 writes

$$S_{T_1} = S_t R_{T_1 - t}.$$

That is

$$\ln S_{T_1} = \ln S_t + \ln R_{T_1-t},$$

so that

$$\ln S_{T_1} \sim N\left(\ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)(T_1 - t), \sigma^2(T_1 - t)\right).$$

At time $t \in [0; T_1)$, the expected value of $\ln S_{T_1}$ writes

$$\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T_1 - t).$$

In a risk-neutral world, the expected value of $\ln S_{T_1}$ writes

$$\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1 - t).$$

Using risk-neutral valuation, the value of the derivative at time $t \in [0; T_1)$ is

$$f = e^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1 - t)\right).$$

b) (1 pt) Since the log return of the stock price is normally distributed, the stock price is continuous everywhere. In particular, we have

$$\lim_{t \to T_1, t < T} \ln S_t = \ln S_{T_1}.$$

From a), we have

$$\begin{aligned}
\lim_{t \to T_1, t<T} f(t) &= e^{-r(T_2-T_1)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1 - T_1)\right) \\
&= e^{-r(T_2-T_1)}\left(\lim_{t \to T_1, t<T} \ln S_t\right) \\
&= e^{-r(T_2-T_1)} \ln S_{T_1} \\
&= \lim_{t \to T_1, t \geq T} f(t).
\end{aligned}$$

So the price, f, of the derivative is continuous at time T_1.

c) (4 pts) If

$$f = \begin{cases} e^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1 - t)\right) & \text{if } t \in [0; T_1) \\ e^{-r(T_2-t)} \ln S_{T_1} & \text{if } t \in [T_1; T_2] \end{cases}$$

then

$$\Theta \equiv \frac{\partial f}{\partial t} = \begin{cases} re^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1-t)\right) - e^{-r(T_2-t)}\left(r - \frac{\sigma^2}{2}\right) & \text{if } t \in [0;T_1) \\ re^{-r(T_2-t)} \ln S_{T_1} & \text{if } t \in [T_1;T_2] \end{cases}$$

$$\Delta \equiv \frac{\partial f}{\partial S} = \begin{cases} \frac{e^{-r(T_2-t)}}{S_t} & \text{if } t \in [0;T_1) \\ 0 & \text{if } t \in [T_1;T_2] \end{cases}$$

$$\Gamma \equiv \frac{\partial^2 f}{\partial S^2} = \begin{cases} -\frac{e^{-r(T_2-t)}}{S_t^2} & \text{if } t \in [0;T_1) \\ 0 & \text{if } t \in [T_1;T_2] \end{cases}$$

So for $t \in [0;T_1)$ the LHS of the Black-Scholes-Merton differential equation writes

$$re^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1-t)\right) - e^{-r(T_2-t)}\left(r - \frac{\sigma^2}{2}\right)$$

$$+rS_t \cdot \frac{e^{-r(T_2-t)}}{S_t} - \frac{\sigma^2}{2}S_t^2 \cdot \frac{e^{-r(T_2-t)}}{S_t^2}$$

$$= re^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1-t)\right) = rf.$$

For $t \in [T_1;T_2]$ the LHS of the Black-Scholes-Merton differential equation writes

$$re^{-r(T_2-t)} \ln S_{T_1} + rS_t \cdot 0 - \frac{\sigma^2}{2}S_t^2 \cdot 0 = re^{-r(T_2-t)} \ln S_{T_1} = rf.$$

Hence the Black-Scholes-Merton differential equation is satisfied.

d) (1 pt) According to answer **a)**, the no-arbitrage current price of the derivative is

$$f = \begin{cases} e^{-r(T_2-t)}\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T_1-t)\right) & \text{if } t \in [0;T_1) \\ e^{-r(T_2-t)} \ln S_{T_1} & \text{if } t \in [T_1;T_2] \end{cases}$$

with $T_1 = 70$, $T_2 = 80$, $t = 30$, $r = 0.025$, $\sigma = 0.18$, and $S_t = 24$, that is

$$f = e^{-0.025(80-30)}\left(\ln 24 + \left(0.025 - \frac{0.18^2}{2}\right)(70-30)\right) \approx 1.01 \text{ €}.$$