Arbitrage and Pricing – Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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Answers can be formulated in English or French.

Exercise 1 (4 pts) There are two periods, $t \in \{0, 1\}$. There are two assets. One non-risky asset (money that can be borrowed or lend) that returns $r = 8\%$ with discrete compounding at time 1. And one risky asset which is a stock of price $S_0 = 3$ at time 0. At date 1, there is either an upward or a downward move. The price of the stock is then either $S_1^u = 4.5$ or $S_1^d = 2$.

Suppose the market price of an European call option on the stock with strike 2.5 euros at time 0 is 0.9 euros.

a) (2 pts) *What should be the non-arbitrage price of the call option at date 0 ?*

b) (2 pts) *Construct an arbitrage portfolio that uses one unit of the call option.*

Exercise 2 (7 pts) A derivative price is currently D_0. It is known that at the end of six months it will be either D_1^u or D_1^d , with $D_1^d < D_1^u$. The risk-free interest rate is $r\%$ per annum with continuous compounding. We consider a derivative instrument on the derivative D, that takes the form of a function $\varphi : \{D_1^d, D_1^u\} \to \mathbb{R}$.

a) (3 pts) *What is the value φ_0 of the derivative φ at the initial date 0 ? Solve the problem using a riskless portfolio that sells one unit of the derivative φ_0 and buy Δ shares of the derivative D_0 at time 0.*

b) (2 pts) *Solve the problem again, but using the equivalent martingale measure approach.*

c) (2 pts) *Numerical application : We consider $r = 0\%$; D is a derivative on a stock with values $S_0 = 5$, $S_1^u = 6$, and $S_1^d = 3$; $D_1^\omega = (S_1^\omega)^2$, $\omega \in \{u, d\}$; and φ is a put option on D_1^ω with strike $K = 25$.*

i) What is the value D_0 (in a risk-neutral world) ?

ii) What is the value φ_0 ?

Exercise 3 (9 pts) Assume that a non-dividend paying stock has an expected return of μ and a volatility of σ with the log return of the stock price being normally distributed. A financial institution has just announced that it will trade a derivative that pays off an euro amount equal to $\ln S_{T_1}$ at time T_2, where S_{T_1} denotes the value of the stock price at time $T_1 > 0$, and time T_2 gives rise to an inherent payment delay, $T_2 - T_1 > 0$. We denote by r the per-period and continuously compounded risk-free interest rate.

a) (3 pts) *What is the price, f, of the derivative at time $t \in [0, T_2]$ according to a risk-neutral valuation ? (Hint : First, start by expressing the price, f, at time $t = T_2$, and $t = T_2 - 1$ before offering the general formulae for any time $t \in [T_1; T_2]$. Second, express the price, f, at any time $t \in [0; T_1)$.)*

b) (1 pt) *Is the price, f, of the derivative continuous at time T_1 ? Prove your statement.*

c) (4 pts) *Verify that your price satisfies the Black-Scholes-Merton differential equation :*

$$\Theta + rS\Delta + \frac{\sigma^2}{2} S^2 \Gamma = rf.$$

d) (1 pt) *Suppose the risk-free interest rate is 2.5%, and the volatility is 18%. The derivative has been originated thirty periods ago and has a remaining life of fifty periods. The inherent payment delay is ten periods. Assume the stock price is currently 24 euros. Give the no-arbitrage current price of the derivative.*