

Arbitrage&Pricing

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Jérôme MATHIS

www.jeromemathis.fr/a-p

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Chapter 3

Chapter 3: Binomial tree with n period Outline

- 1 Introduction
- 2 Binomial Trees: Two-Step
- 3 Binomial tree: generalization
- 4 Conclusion

Chapter 3: Binomial tree with n period Outline

- 1 Introduction
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Introduction

- The one-period model is often too simple for practical purpose.
 - ▶ An individual investor has approximately 50 years of adult life when he is making choices over savings, investment and consumption.
 - ★ If important investment decisions are taken every five years, we need at least a 10-period model.
 - ▶ Professional investors trade even more frequently.
 - ★ A trader on a stock exchange may adjust his portfolio several times a day resulting in more than 500 investment decisions a month.

- The aim of this chapter is to introduce the techniques to asset pricing in a dynamic framework.
 - ▶ We use a simple set-up with the European call option as a focus asset in a discrete-time model:
 - ★ to illustrate the backward recursive pricing procedure; and
 - ★ to recover the option price as an unconditional expectation under risk-neutral probabilities.

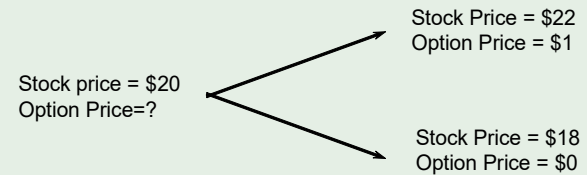
Chapter 3: Binomial tree with n period Outline

- 1 Introduction
- 2 **Binomial Trees: Two-Step**
 - Generalization
 - A Put Option Example
 - Delta
- 3 Binomial tree: generalization
- 4 Conclusion

- Consider Example D of Chapter 2:

Example

A 3-month call option on the stock has a strike price of 21.



Binomial Trees: Two-Step

- Applying the formula

$$C_0 = \frac{1}{R} \left(\frac{R-d}{u-d} C_1^u + \frac{u-R}{u-d} C_1^d \right)$$

to the 3 months risk-free interest rate of 3.05%, we found the initial price of the option:

$$C_0 = \frac{1}{1.0305} \left(\frac{1.0305 - 0.9}{1.1 - 0.9} \right) \simeq 0.633$$

Binomial Trees: Two-Step

- We also obtained the same initial price of the option using a risk-neutral valuation.
- Indeed, by denoting q the probability that gives a return on the stock equal to the risk-free rate:

$$S_0(1+r) = S_1^u q + S_1^d(1-q).$$

- The value of the option is

$$C_0 = \frac{C_1^u q + C_1^d(1-q)}{1+r}$$

- So that in Example $D^{(5)}$ we obtained

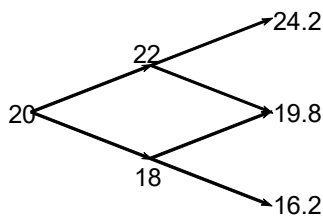
$$20(1.0305) = 22q + 18(1-q).$$

so that $q = 0.6525$. And

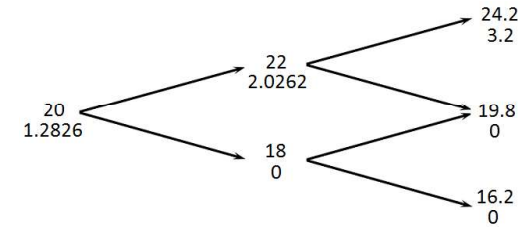
$$C_0 = \frac{1 \times 0.6525 + 0(1 - 0.6525)}{1.0305} \simeq 0.6332.$$

Binomial Trees: Two-Step

- Let us extend this example to a two-step binomial tree.
- Assume the stock price starts at \$20 and in each of two time steps may go up by 10% or down by 10%.
 - ▶ Each time step is 3 months long and 3 months risk-free interest rate of 3.05%.
 - ▶ We consider a 6-month option with a strike price of \$21.



Binomial Trees: Two-Step



- When the stock price is 22, the option price is

$$\frac{0.6525 \times 3.2 + 0.3475 \times 0}{1.0305} = 2.0262$$

- When the stock price is 18, the option price is zero, because it leads to two nodes where the option price is zero.

- The initial option price is

$$\frac{0.6525 \times 2.0262 + 0.3475 \times 0}{1.0305} = 1.283$$

Binomial Trees: Two-Step Generalization

- Suppose that the risk-free interest rate is r , with continuous compounding, and the length of the time step is Δt years.
- We have

$$C_0 = [qC_1^u + (1-q)C_1^d]e^{-r\Delta t}$$

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

$$C_1^u = [qC_2^{uu} + (1-q)C_2^{ud}]e^{-r\Delta t}$$

and

$$C_1^d = [qC_2^{du} + (1-q)C_2^{dd}]e^{-r\Delta t}$$

- So, when $C_2^{du} = C_2^{ud}$ we obtain

$$C_0 = [q^2 C_2^{uu} + 2q(1-q)C_2^{ud} + (1-q)^2 C_2^{dd}]e^{-2r\Delta t}$$

Binomial Trees: Two-Step

A Put Option Example

Question

Consider a 2-year European put with a strike price of \$52 on a stock whose current price is \$50.

We suppose that there are two time steps of 1 year, and in each time step the stock price either moves up by 20% or moves down by 20%.

We also suppose that the risk-free interest rate is 5%.

What is the initial price of the option?

Binomial Trees: Two-Step

A Put Option Example

Solution

Binomial Trees: Two-Step

A Put Option Example

Solution

Binomial Trees: Two-Step

A Put Option Example

Solution

Binomial Trees: Two-Step

Delta

Definition

Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.

- It is the number of units of the stock we should hold for each option shorted in order to create a riskless portfolio.
 - ▶ It is the same as the Δ introduced earlier in this and previous chapters.
- The construction of a riskless portfolio is sometimes referred to as **delta hedging**.
- The delta of a call option is positive, whereas the delta of a put option is negative.
- The value of Δ varies from node to node.
 - ▶ E.g., when the stock price changes from \$18 to \$22, and the option price changes from \$0 to \$1, we have $\Delta = \frac{1-0}{22-18} = 0.25$.

Chapter 3: Binomial tree with n period

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 - Basic notions on Probability
 - Setup
 - Simple portfolio strategies
 - Arbitrage and risk-neutral probability
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Binomial tree: generalization

Basic notions on Probability

Definition

A **filtration** is a sequence of σ -algebra $(\mathcal{F}_k)_{1 \leq k \leq n}$, such that each σ -algebra in the sequence contains all the sets contained by the previous σ -algebra. Formally, $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}, \forall k < n$.

- A filtration models the evolution of information through time. So for example, if it is known by time k whether or not an event, E , has occurred, then we have $E \in \mathcal{F}_k$.

Definition

Let $\mathcal{F} := (\mathcal{F}_k)_{1 \leq k \leq n}$ be a filtration. The stochastic process $(X_k)_{1 \leq k \leq n}$ is **\mathcal{F} -adapted**, if X_k is \mathcal{F}_k -measurable for each $k \leq n$.

- The idea is that the value of X_k is known at time k when the information represented by \mathcal{F}_k is known.

Binomial tree: generalization

Basic notions on Probability

Proposition (3.1)

If the stochastic process $(X_k)_{1 \leq k \leq n}$ is \mathcal{F} -adapted, then X_i is \mathcal{F}_k -measurable for any $i \leq k$.

Proof.

Straightforward.

If the stochastic process $(X_k)_{1 \leq k \leq n}$ is \mathcal{F} -adapted then X_i is \mathcal{F}_i -measurable.

Since $\mathcal{F} := (\mathcal{F}_k)_{1 \leq k \leq n}$ is a filtration then $\mathcal{F}_i \subseteq \mathcal{F}_k, \forall i \leq k$, with $k \leq n$.

So, X_i is \mathcal{F}_k -measurable. □

Definition

The **natural filtration** of the stochastic process $(X_k)_{1 \leq k \leq n}$ is given by the smallest filtration \mathcal{F} for which $(X_k)_{1 \leq k \leq n}$ is \mathcal{F} -adapted.

We denote it by $\mathcal{F}^X := (\mathcal{F}_k^X)_{1 \leq k \leq n}$, with \mathcal{F}_k^X the σ -algebra generated by X_k .

I.e., $\mathcal{F}_k^X := \sigma(X_1, X_2, \dots, X_k)$.

Definition (new)

A stochastic process $M := (M_k)_{1 \leq k \leq n}$ is a \mathcal{F} -**martingale** under \mathbb{P} if M is \mathcal{F} -adapted,

$$\mathbb{E}^{\mathbb{P}}[|M_k|] < +\infty, \text{ for any } k \leq n$$

and

$$\mathbb{E}^{\mathbb{P}}[M_{k+1} | \mathcal{F}_k] = M_k.$$

- If the previous equality is replaced with \leq the process tends to go down and is called a *supermartingale*. If the previous equality is replaced with \geq the process tends to go up and is called a *submartingale*.
- So, a supermartingale (resp. submartingale) is a losing (resp. winning) game.

- If $M := (M_k)_{1 \leq k \leq n}$ is a \mathcal{F} -**martingale** under \mathbb{P} then

$$\mathbb{E}^{\mathbb{P}}[M_k | \mathcal{F}_i] = M_i, \text{ for any } i \leq k$$

and in particular, we have

$$\mathbb{E}^{\mathbb{P}}[M_k] = M_0.$$

- We extend the model of the previous chapter to n periods.
- We consider an interval of time $[0, T]$ divided into n periods:
 $0 = t_0 \leq t_1 \leq \dots \leq t_n = T$.
- There are two assets:

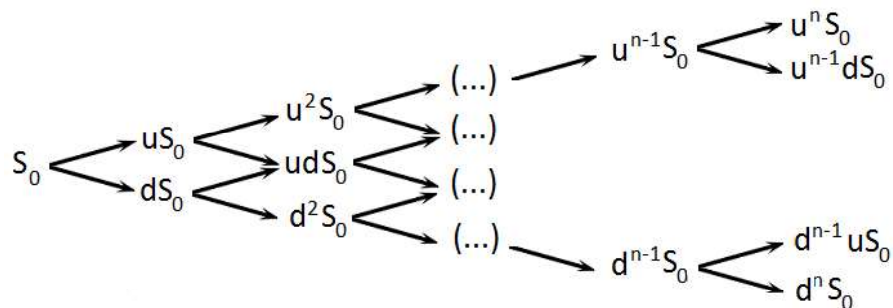
- ▶ A *non-risky asset* S_t^0 :

$$1 \rightarrow (1+r) \rightarrow (1+r)^2 \rightarrow \dots \rightarrow (1+r)^n$$

- ▶ A *risky asset* S_t that evolves according to the following Table

Binomial tree: generalization

Set-up



- The order of occurrence of u and d 's does not count. So, the tree recombines (e.g., $du^2S_0 = uduS_0 = u^2dS_0$). At time t the asset may then only take $t + 1$ values. (If the order would have count we would have obtained 2^t values.)

Binomial tree: generalization

Set-up

- At date t Nature selects $\omega_t \in \{\omega_t^u, \omega_t^d\}$. So

$$\Omega := \{(\omega_1, \omega_2, \dots, \omega_n) \mid \forall i \in \{0, 1, \dots, n\} \text{ we have } \omega_i = \omega_i^u \text{ or } \omega_i = \omega_i^d\}.$$

We assume that the probability of occurrence of u is time-invariant:

$$\mathbb{P}(\omega_i = \omega_i^u) = p$$

and

$$\mathbb{P}(\omega_i = \omega_i^d) = 1 - p.$$

Binomial tree: generalization

Set-up

- So

$$\mathbb{P}(\omega_1, \omega_2, \dots, \omega_n) = p^{\#\{i \in \{0, 1, \dots, n\} \mid \omega_i = \omega_i^u\}} \times (1 - p)^{\#\{i \in \{0, 1, \dots, n\} \mid \omega_i = \omega_i^d\}}.$$

- The value of the asset at time t , can be written as

$$S_t = S_0 \prod_{k=0}^t Y_k$$

with $(Y_k)_{k=0, \dots, n} : \Omega^{n+1} \mapsto \{u, d\}^{n+1}$ being a collection of random variables i.i.d., with Y_k is realized at time k , and takes the value u with probability p and d with probability $(1 - p)$.

Binomial tree: generalization

Set-up

- So we have

$$\mathbb{P}(Y_i = u) = \mathbb{P}(\omega_i = \omega_i^u) = p$$

and

$$\mathbb{P}(Y_i = d) = \mathbb{P}(\omega_i = \omega_i^d) = 1 - p.$$

Binomial tree: generalization

Set-up

- The information available at time t_i is given by the filtration $(\mathcal{F}_{t_k})_{1 \leq k \leq i}$, with

$$\mathcal{F}_{t_i} := \sigma(\omega_1, \omega_2, \dots, \omega_i) = \sigma(Y_1, Y_2, \dots, Y_i) = \sigma(S_{t_1}, S_{t_2}, \dots, S_{t_i}).$$

Definition (Mathematics)

In our market, a **derivative** is a random variable that is \mathcal{F}_T -measurable.

- So, a derivative takes the form of a function $\phi(S_{t_1}, S_{t_2}, \dots, S_{t_i})$.

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Binomial tree: generalization

Simple portfolio strategies

Definition

A **simple portfolio strategy** consists in an initial amount of cash x and a stochastic process $\Delta := (\Delta_k)_{0 \leq k \leq n-1}$ which is \mathcal{F} -adapted.

We denote this strategy by the pair (x, Δ) and its value at date t_i by $X_{t_i}^{x, \Delta}$.

- A simple portfolio strategy consists in using a part of an initial amount of cash x to buy (at the initial date) the risky asset in quantity Δ_0 , and to invest the other part of x in a non-risky asset.
 - Then at date t_i we invest into the risky asset in quantity Δ_i .
 - The process is \mathcal{F} -adapted because the amount of investment at date t_i is determined using the information available at date t_i .
 - This simple portfolio strategy is self-financing.

Binomial tree: generalization

Simple portfolio strategies

- Between period t_i and t_{i+1} , the portfolio takes the form of Δ_i units of risky asset and $\frac{X_{t_i}^{x, \Delta} - \Delta_i S_{t_i}}{(1+r)^i}$ units of non-risky asset. So, the value of the portfolio at time t_i is given by

$$X_{t_i}^{x, \Delta} = \Delta_i S_{t_i} + \frac{X_{t_i}^{x, \Delta} - \Delta_i S_{t_i}}{(1+r)^i} (1+r)^i.$$

- Since the strategy is self-financing, no money is withdrawn nor inserted during the time interval $[t_i, t_{i+1})$, so we have

$$X_{t_{i+1}}^{x, \Delta} = \Delta_i S_{t_{i+1}} + \frac{X_{t_i}^{x, \Delta} - \Delta_i S_{t_i}}{(1+r)^i} (1+r)^{i+1}.$$

Binomial tree: generalization

Simple portfolio strategies

- Let \tilde{Z} denotes the **current value** of the variable Z at date $t = 0, 1, \dots, n$.

► So, the current value of the portfolio X at date t_i writes as

$$\tilde{X}_{t_i}^{x,\Delta} := \frac{X_{t_i}^{x,\Delta}}{(1+r)^i}$$

► and the current value of the risky asset at date t_i writes as

$$\tilde{S}_{t_i} := \frac{S_{t_i}}{(1+r)^i}.$$

Binomial tree: generalization

Simple portfolio strategies

- So we have

$$\begin{aligned} \tilde{X}_{t_{i+1}}^{x,\Delta} &= \frac{X_{t_{i+1}}^{x,\Delta}}{(1+r)^{i+1}} \\ &= \frac{\Delta_i S_{t_{i+1}} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i} (1+r)^{i+1}}{(1+r)^{i+1}} \\ &= \Delta_i \tilde{S}_{t_{i+1}} + \frac{X_{t_i}^{x,\Delta} - \Delta_i S_{t_i}}{(1+r)^i} \\ &= \Delta_i \tilde{S}_{t_{i+1}} + \left(\tilde{X}_{t_i}^{x,\Delta} - \Delta_i \tilde{S}_{t_i} \right). \end{aligned}$$

Binomial tree: generalization

Simple portfolio strategies

- From

$$\begin{aligned} \tilde{X}_{t_i}^{x,\Delta} &= \tilde{X}_{t_i}^{x,\Delta} + \left(\Delta_i \tilde{S}_{t_i} - \Delta_i \tilde{S}_{t_i} \right) \\ &= \Delta_i \tilde{S}_{t_i} + \left(\tilde{X}_{t_i}^{x,\Delta} - \Delta_i \tilde{S}_{t_i} \right) \end{aligned}$$

we obtain the self-financing condition

$$\tilde{X}_{t_{i+1}}^{x,\Delta} - \tilde{X}_{t_i}^{x,\Delta} = \Delta_i \left(\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i} \right). \quad (1)$$

- This condition can be rewritten as

$$\tilde{X}_{t_{i+1}}^{x,\Delta} = x + \sum_{k=0}^i \Delta_k \left(\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k} \right).$$

Binomial tree: generalization

Arbitrage and risk-neutral probability

Definition

A **simple arbitrage** is a simple portfolio strategy that gives to a portfolio no value at time $t = 0$ and a value at time $T = t_n$ which is strictly positive with positive probability and is never negative. Formally, it is a pair $(x = 0, \Delta)$ with $\Delta \in \mathbb{R}^n$ such that

$$X_T^{0,\Delta} \geq 0 \text{ and } \mathbb{P}(X_T^{0,\Delta} > 0) > 0.$$

Definition

We say that there is **no simple arbitrage opportunity (NAO')** if

$$\forall \Delta \in \mathbb{R}^n \{ X_T^{0,\Delta} \geq 0 \implies X_T^{0,\Delta} = 0 \text{ } \mathbb{P} - \text{a.s.} \}$$

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proposition (3.2)

If NAO' then $d < 1 + r < u$.

Proof.

We proceed by contradiction. Assume NAO' and $d \geq 1 + r$.

Consider the following simple arbitrage strategy:

- buy one unit of the risky asset; and
- sell the equivalent amount of the non risky asset in period $t = 0$;
- then resell the unit of the risky asset at time t_1 ; and
- invest it into the non-risky asset until period T
- (i.e., $x = 0$, $\Delta_0 = 1$, and $\Delta_i = 0$ for any $i \geq 1$). (...)

□

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

Since S_{t_1} can only take two values (either uS_0 or dS_0), the portfolio value at time T is either

$$(1 + r)^n S_0 \left(\frac{u}{(1 + r)} - 1 \right) > 0$$

or

$$(1 + r)^n S_0 \left(\frac{d}{(1 + r)} - 1 \right) \geq 0$$

which contradicts NAO', since both values occur with strictly positive probabilities (resp. p and $(1 - p)$). (...)

□

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

Such a strategy is deterministic so it is \mathcal{F} -adapted. At date $T = t_n$ the portfolio value is given by:

$$\begin{aligned} \tilde{X}_T^{0,\Delta} &= 0 + \sum_{k=0}^{n-1} \Delta_k (\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k}) \\ &= \tilde{S}_{t_1} - \tilde{S}_{t_0} \end{aligned}$$

(...)

□

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

Similarly, we obtain a contradiction by assuming $u \leq 1 + r$ and by considering the simple arbitrage strategy that consists in:

- selling one unit of the risky asset; and
- buying the equivalent amount of the non risky asset in period $t = 0$;
- i.e., $x = 0$, $\Delta_0 = -1$, and $\Delta_i = 0$ for any $i \geq 1$.

□

Binomial tree: generalization

Arbitrage and risk-neutral probability

- Consider the following probability on Ω :

$$\mathbb{Q}(\omega_1, \omega_2, \dots, \omega_n) = q^{\#\{i \in \{1, \dots, n\} | \omega_i = \omega_i^u\}} \times (1 - q)^{\#\{i \in \{1, \dots, n\} | \omega_i = \omega_i^d\}}$$

with

$$q := \frac{(1 + r) - d}{u - d}.$$

- We then have

$$\mathbb{Q}(S_t = uS_{t-1}) = \mathbb{Q}(Y_i = u) = q$$

and

$$\mathbb{Q}(S_t = dS_{t-1}) = \mathbb{Q}(Y_i = d) = 1 - q.$$

- Let us show that \mathbb{Q} is a risk-neutral probability measure.

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

Moreover, we have

$$\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t] = \frac{\mathbb{E}^{\mathbb{Q}}[S_t]}{(1 + r)^t} < +\infty, \text{ for any } i \leq n$$

and (...)



Binomial tree: generalization

Arbitrage and risk-neutral probability

- By definition (see chap. 2), a risk-neutral probability measure or equivalent martingale measure (EMM) is a probability measure \mathbb{Q} which is equivalent to \mathbb{P} and for which any simple strategy expressed in current value is a martingale.

Proposition (3.3)

$\tilde{S} := \left(\tilde{S}_t \right)_{t \in \{1, 2, \dots, n\}}$ is a \mathcal{F} -martingale under \mathbb{Q} .

Proof.

Clearly, $\tilde{S}_t := \frac{S_t}{(1+r)^t}$ is \mathcal{F}_t -measurable for each $i \leq n$, so \tilde{S} is \mathcal{F} -adapted.(...)



Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t+1} | \mathcal{F}_t] &= \frac{qu\tilde{S}_t + (1 - q)d\tilde{S}_t}{1 + r} \\ &= \frac{1}{1 + r} \left(\frac{(1 + r) - d}{u - d} u\tilde{S}_t + \left(1 - \frac{(1 + r) - d}{u - d} \right) d\tilde{S}_t \right) \\ &= \frac{1}{1 + r} \left(\frac{(1 + r) - d}{u - d} u + \frac{u - (1 + r)}{u - d} d \right) \tilde{S}_t \\ &= \frac{1}{1 + r} \left(\frac{(1 + r)(u - d)}{u - d} \right) \tilde{S}_t = \tilde{S}_t. \end{aligned}$$



Binomial tree: generalization

Arbitrage and risk-neutral probability

- The following result states that if the current values of the standard assets are martingale under a given probability then it is so of the current value of any simple portfolio strategy.

Proposition (3.4)

The current value $\tilde{X}^{x,\Delta}$ of any simple portfolio strategy (x, Δ) is a \mathcal{F} -martingale under \mathbb{Q} .

Proof.

Clearly, $\tilde{X}^{x,\Delta}$ is \mathcal{F} -adapted. Moreover, we have

$$\mathbb{E}^{\mathbb{Q}}[|\tilde{X}_i^{x,\Delta}|] < +\infty, \text{ for any } i \leq n.$$

(...)

□

Binomial tree: generalization

Arbitrage and risk-neutral probability

Theorem (3.5)

If $d < R < u$ then there is an equivalent martingale measure \mathbb{Q} .

Proof.

According to the previous result we know that \mathbb{Q} is a probability measure for which any simple strategy expressed in current value is a martingale. Moreover, \mathbb{Q} is equivalent to \mathbb{P} since $d < R < u$ implies that $q \in (0, 1)$ and that $\mathbb{Q}(\omega_1, \omega_2, \dots, \omega_n) > 0$ for every $(\omega_1, \omega_2, \dots, \omega_n) \in \Omega$. □

The value at date t_i of any simple portfolio strategy writes as

$$X_{t_i}^{x,\Delta} = \frac{\mathbb{E}^{\mathbb{Q}}[X_T^{x,\Delta} | \mathcal{F}_{t_i}]}{(1+r)^{n-i}}.$$

Binomial tree: generalization

Arbitrage and risk-neutral probability

Proof.

Now it suffices to show that

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}_{t_{i+1}}^{x,\Delta} - \tilde{X}_{t_i}^{x,\Delta} | \mathcal{F}_{t_i}] = 0.$$

From (1) we have

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}_{t_{i+1}}^{x,\Delta} - \tilde{X}_{t_i}^{x,\Delta} | \mathcal{F}_{t_i}] = \mathbb{E}^{\mathbb{Q}}[\Delta_i (\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i}) | \mathcal{F}_{t_i}]$$

so

$$\mathbb{E}^{\mathbb{Q}}[\tilde{X}_{t_{i+1}}^{x,\Delta} - \tilde{X}_{t_i}^{x,\Delta} | \mathcal{F}_{t_i}] = \Delta_i \mathbb{E}^{\mathbb{Q}}[\tilde{S}_{t_{i+1}} - \tilde{S}_{t_i} | \mathcal{F}_{t_i}] = 0$$

where the last equality comes from the previous Proposition. □

Binomial tree: generalization

Arbitrage and risk-neutral probability

- Hence, if we are able to hedge a derivative, NAO implies that the value of the hedging portfolio at date t_i is given by the expected current value of its final value under the risk-neutral probability.
- Before exploiting this idea, let us state the following result.

Proposition (3.6)

If there is an equivalent martingale measure \mathbb{Q} then NAO' holds.

Proof.

As in the previous chapter, let $\Delta \in \mathbb{R}^n$ such that $X_T^{0,\Delta} \geq 0$.

Since \mathbb{Q} is an equivalent martingale measure, we have

$$\mathbb{E}^{\mathbb{Q}} \left[X_T^{x=0,\Delta} \right] = x = 0.$$

Which means that $X_T^{0,\Delta}$ is a random variable that is positive and whose expected value is zero.

This variable is then equal to zero \mathbb{Q} – a.s.

Finally, since \mathbb{Q} is equivalent to \mathbb{P} we obtain $\mathbb{P} \left(X_T^{0,\Delta} > 0 \right) = 0$. □

- From Proposition 3.2

$$NAO' \implies d < R < u$$

- From Theorem 3.5

$$d < R < u \implies \text{there is an equivalent martingale measure}$$

- From Proposition 3.6 we have

$$\text{there is an equivalent martingale measure} \implies NAO'$$

- Hence we obtain

$$\begin{aligned} NAO' &\iff d < R < u \\ &\iff \text{there is an equivalent martingale measure.} \end{aligned}$$

- Saying that

“the current values of every standard asset is martingale under \mathbb{Q} ”
is then equivalent to say that

“the current value of every simple portfolio strategy is martingale under \mathbb{Q} ”.

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Binomial tree: generalization

Hedging derivative

Theorem (3.7)

In our market, every derivative is replicable by using a simple portfolio strategy (x, Δ) .

Question

What is the form of (x, Δ) ?

- We are looking at for a simple portfolio strategy (x, Δ) replicating a derivative of value C_T at date T . Since C_T is \mathcal{F}_{t_n} -adapted, the value of the derivative takes the form of a function $\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})$ so (x, Δ) has to satisfy

$$X_{t_n}^{x, \Delta} = \phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}).$$

Binomial tree: generalization

Hedging derivative

- According to Proposition 3.4, the current value of every simple portfolio strategy is martingale under the EMM \mathbb{Q} , so the value $X_{t_k}^{x, \Delta}$ of the replicating portfolio at date t_k satisfies

$$X_{t_k}^{x, \Delta} = \frac{\mathbb{E}^{\mathbb{Q}} [X_{t_n}^{x, \Delta} | \mathcal{F}_{t_k}]}{(1+r)^{n-k}} = \frac{\mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]}{(1+r)^{n-k}}.$$

- So the initial amount of our replicating portfolio has to be

$$x := \frac{\mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})]}{(1+r)^n}. \quad (2)$$

Binomial tree: generalization

Hedging derivative

- Now, since $\frac{1}{(1+r)^{n-k}} \mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]$ is a random variable which is \mathcal{F}_{t_k} -measurable it can be rewritten as a function $V_k(S_{t_1}, S_{t_2}, \dots, S_{t_k})$ where $V_k(\cdot)$ is deterministic. Let

$$V_k(S_{t_1}, S_{t_2}, \dots, S_{t_k}) := \frac{\mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]}{(1+r)^{n-k}}. \quad (3)$$

- In the previous chapter which introduces the model with one period, we have seen that the quantity of the risky asset Δ of the replicating portfolio looks like the variation of the value of the derivative induced by the variation of the underlying asset.

Hedging derivative

- In the proof of Theorem, we shall take $\Delta := (\Delta_k)_{k \in \{1, 2, \dots, n\}}$ satisfying that for any $k \in \{1, 2, \dots, n\}$

$$\Delta_k := \frac{V_{k+1}(S_{t_1}, S_{t_2}, \dots, S_{t_k}, uS_{t_k}) - V_{k+1}(S_{t_1}, S_{t_2}, \dots, S_{t_k}, dS_{t_k})}{uS_{t_k} - dS_{t_k}}. \quad (4)$$

- Observe that for any $k \in \{1, 2, \dots, n\}$, Δ_k is \mathcal{F}_{t_k} -measurable as a function of $(S_{t_1}, S_{t_2}, \dots, S_{t_k})$. So, Δ is \mathcal{F} -adapted and (x, Δ) is indeed a simple portfolio strategy.
- Now, let us establish the proof of Theorem 3.7 according to which the simple portfolio strategy (x, Δ) with Δ satisfying (4) replicates our derivative.

Binomial tree: generalization

Hedging derivative

Proof.

We have to show that

$$X_{t_n}^{x,\Delta} = \phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) = V_n(S_{t_1}, \dots, S_{t_n}).$$

Let us proceed by induction. Let $P(k)$, $k \in \{1, 2, \dots, n\}$ be the following statement:

$$X_{t_k}^{x,\Delta} = V_k(S_{t_1}, \dots, S_{t_k})$$

Clearly, $P(0)$ is true. Indeed, we have $X_{t_0}^{x,\Delta} = x$ and by (2)

$$x := \frac{\mathbb{E}^{\mathbb{Q}}[\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})]}{(1+r)^n} = \frac{\mathbb{E}^{\mathbb{Q}}[\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_0}]}{(1+r)^{n-0}}$$

which by (3) correspond to $V_0(S_{t_0})$. (...)

□

Binomial tree: generalization

Hedging derivative

Proof.

Now using that for any martingale Z that is \mathcal{F} -measurable we have for any k and s such that $k + s < n$

$$\mathbb{E}[Z_{t_n} | \mathcal{F}_{t_k}] = \mathbb{E}[\mathbb{E}[Z_{t_n} | \mathcal{F}_{t_{k+s}}] | \mathcal{F}_{t_k}]$$

we obtain

$$\begin{aligned} X_{t_k}^{x,\Delta} &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{Q}} \left[\frac{\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})}{(1+r)^{n-(k+1)}} \middle| \mathcal{F}_{t_{k+1}} \right] \middle| \mathcal{F}_{t_k} \right] \\ &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}} [V_{k+1}(S_{t_1}, \dots, S_{t_{k+1}}) | \mathcal{F}_{t_k}]. \end{aligned}$$

(...)

□

Binomial tree: generalization

Hedging derivative

Proof.

Assume $P(k)$ is true. Let us show that $P(k+1)$ is true.

$P(k)$ writes as

$$\begin{aligned} X_{t_k}^{x,\Delta} &= V_k(S_{t_1}, \dots, S_{t_k}) \\ &= \frac{\mathbb{E}^{\mathbb{Q}}[\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]}{(1+r)^{n-k}} \\ &= \frac{1}{(1+r)} \frac{\mathbb{E}^{\mathbb{Q}}[\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]}{(1+r)^{n-k-1}} \\ &= \frac{1}{(1+r)} \mathbb{E}^{\mathbb{Q}} \left[\frac{\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})}{(1+r)^{n-(k+1)}} \middle| \mathcal{F}_{t_k} \right]. \end{aligned}$$

(...)

□

Binomial tree: generalization

Hedging derivative

Proof.

So,

$$\begin{aligned} X_{t_k}^{x,\Delta} &= \frac{\mathbb{E}^{\mathbb{Q}} \left[\begin{aligned} &V_{k+1}(S_{t_1}, \dots, S_{t_k}, u S_{t_k}) \mathbf{1}_{\{Y_{t_{k+1}}=u\}} \\ &+ V_{k+1}(S_{t_1}, \dots, S_{t_k}, d S_{t_k}) \mathbf{1}_{\{Y_{t_{k+1}}=d\}} \end{aligned} \middle| \mathcal{F}_{t_k} \right]}{(1+r)} \\ &= \frac{\mathbb{Q}(Y_{t_{k+1}}=u) V_{k+1}(S_{t_1}, \dots, S_{t_k}, u S_{t_k}) + \mathbb{Q}(Y_{t_{k+1}}=d) V_{k+1}(S_{t_1}, \dots, S_{t_k}, d S_{t_k})}{(1+r)} \end{aligned}$$

□

Binomial tree: generalization

Hedging derivative

Proof.

That is

$$X_{t_k}^{x,\Delta} = \frac{[qV_{k+1}(S_{t_1}, \dots, S_{t_k}, uS_{t_k}) + (1-q)V_{k+1}(S_{t_1}, \dots, S_{t_k}, dS_{t_k})]}{(1+r)} \quad (5)$$

Now, from (1) we have

$$\tilde{X}_{t_{k+1}}^{x,\Delta} = \tilde{X}_{t_k}^{x,\Delta} + \Delta_k (\tilde{S}_{t_{k+1}} - \tilde{S}_{t_k})$$

so

$$\frac{X_{t_{k+1}}^{x,\Delta}}{(1+r)^{k+1}} = \frac{X_{t_k}^{x,\Delta}}{(1+r)^k} + \Delta_k \left(\frac{S_{t_{k+1}}}{(1+r)^{k+1}} - \frac{S_{t_k}}{(1+r)^k} \right)$$

(...)

Binomial tree: generalization

Hedging derivative

Proof.

By replacing $S_{t_{k+1}}$ with $Y_{k+1}S_{t_k}$ and q with $\frac{1+r-d}{u-d}$ we have

$$X_{t_{k+1}}^{x,\Delta} = V_{k+1}(S_{t_1}, \dots, S_{t_k}, uS_{t_k}) \frac{Y_{k+1} - d}{u - d} + V_{k+1}(S_{t_1}, \dots, S_{t_k}, dS_{t_k}) \frac{u - Y_{k+1}}{u - d}.$$

Since Y_{k+1} can only take the value u and d we obtain

$$X_{t_{k+1}}^{x,\Delta} = V_{k+1}(S_{t_1}, \dots, S_{t_k}, Y_{k+1}S_{t_k}) = V_{k+1}(S_{t_1}, \dots, S_{t_k}, S_{t_{k+1}})$$

which is $P(k+1)$. (...)

Binomial tree: generalization

Hedging derivative

Proof.

which rewrites as

$$X_{t_{k+1}}^{x,\Delta} = X_{t_k}^{x,\Delta} (1+r) + \Delta_k (S_{t_{k+1}} - (1+r)S_{t_k})$$

Using (5) and (4) we obtain

$$\begin{aligned} X_{t_{k+1}}^{x,\Delta} &= qV_{k+1}(S_{t_1}, \dots, S_{t_k}, uS_{t_k}) + (1-q)V_{k+1}(S_{t_1}, \dots, S_{t_k}, dS_{t_k}) \\ &+ \frac{V_{k+1}(S_{t_1}, S_{t_2}, \dots, S_{t_k}, uS_{t_k}) - V_{k+1}(S_{t_1}, S_{t_2}, \dots, S_{t_k}, dS_{t_k})}{uS_{t_k} - dS_{t_k}} \\ &\times (S_{t_{k+1}} - (1+r)S_{t_k}). \end{aligned}$$

(...)

Binomial tree: generalization

Hedging derivative

Proof.

Since every derivative is replicable, under NAO, a derivative of final value

$$C_T = \phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})$$

has a value at date t_k given by

$$C_{t_k} = \frac{1}{(1+r)^{n-k}} \mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n}) | \mathcal{F}_{t_k}]$$

and in particular at date 0

$$C_0 = \frac{1}{(1+r)^n} \mathbb{E}^{\mathbb{Q}} [\phi(S_{t_1}, S_{t_2}, \dots, S_{t_n})].$$

Binomial tree: generalization

Hedging derivative

- It means that the derivative price at any date can be obtained by backward induction.
 - ▶ we can treat each binomial step separately and work back from the end of the life of the option to the beginning to obtain the current value of the option.
- The following result extends Proposition 2.6 of Chapter 2 to our setup.

Proposition (3.8)

If every asset is replicable with a simple portfolio strategy (complete market) then the equivalent martingale measure is unique.

Proof.

The proof is the one of Proposition 2.6. □

Chapter 3: Binomial tree with n period

Outline

- 1 Introduction
- 2 Binomial Trees: Two-Step
- 3 Binomial tree: generalization
- 4 Conclusion

Conclusion

- The binomial model with n periods produces similar results to the model with one period:
 - ▶ the derivative price does not depend the probabilities of up, p , and down, $(1 - p)$, movements in the stock price at each node of the tree.
 - ▶ the derivative price is the expected current value, expressed with the equivalent martingale measure \mathbb{Q} , of its future value.
 - ▶ the quantity Δ of the risky asset in the replicative portfolio measures how the derivative price moves with the underlying asset price.
- When stock price movements are governed by a multistep binomial tree, we can use backward induction to deduce the initial option price from the final option price.

Conclusion

- We can assume the world is risk-neutral when valuing an option.
 - ▶ No-arbitrage arguments and risk-neutral valuation are equivalent and lead to the same option prices.
- The delta of a stock option, Δ , considers the effect of a small change in the underlying stock price on the change in the option price.
 - ▶ It is the ratio of the change in the option price to the change in the stock price.
 - ▶ For a riskless position, an investor should buy Δ shares for each option sold.
 - ▶ An inspection of a typical binomial tree shows that delta changes during the life of an option.
 - ▶ This means that to hedge a particular option position, we must change our holding in the underlying stock periodically.