Industrial Organization

Master Industries de Réseau et Economie Numérique (IREN) - 2024/2025

Chapter 1: Static Models of Oligopoly

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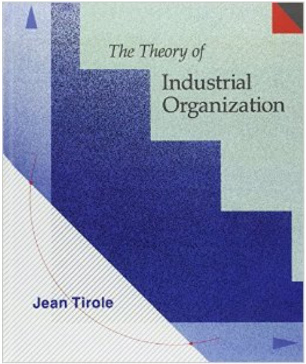
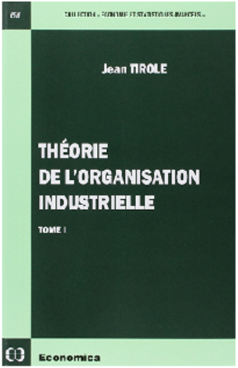
Chapter 1





English ed.: MIT; French ed.: Economica

Introduction
Jean Tirole (Nobel 2014)

Affiliation: Toulouse School of Economics (TSE), Toulouse, France

- **Prize motivation:** "for his analysis of market power and regulation"
- **Field:** industrial organization, microeconomics

- Questions:
  - What is the price on a given market?
  - What are the profits?
  - What is the social surplus?
- Answers. It depends on:
  - How many firms are on the market
    - ⋆ Monopoly, duopoly, oligopoly, ... , atomless firms.
  - Whether firms are competing on prices or on quantity.
  - Whether there are capacity constraints, decreasing returns to scale, ...
  - Whether there is a temporal dimension, product differenciation, ...

- You already know that:
  - profit is maximal under monopoly
    - ⋆ price is chosen such that profit is maximal
  - profit is minimal under pure and perfect competition
    - ⋆ price equals marginal cost.

Joseph Louis François Bertrand (1822-1900)

- $N = \{1, 2\}$: Two firms produce goods that are perfect substitutes in the consumers' utility functions.
- The market demand function is

$$q = D(p)$$

and the demand for the output of firm i, $i \in N$, denoted as D_i, is

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j \\ \frac{D(p_i)}{2} & \text{if} \quad p_i = p_j \\ 0 & \text{otherwise} \end{cases}$$

- Each firm incurs a cost c per unit of production.
- So the profit of firm i is:

$$\pi_i(p_i, p_j) = (p_i - c) D_i(p_i, p_j)$$

Proposition (Bertrand (1883))

The unique equilibrium has the two firms price at marginal cost and do not make profits.

Proof.

Assume (p_1^*, p_2^*) is an equilibrium. Let us show that $p_1^* = p_2^* = c$
Assume $p_k^* = c$. By charging $p_j^* \neq c$, firm $j \neq k$ makes either zero profits (if $p_j^* > c = p_k^*$) or negative profits (if $p_j^* < c = p_k^*$). By charging $p_j^* = c$ firm j makes zero profits and there is no profitable deviation. □

Question

Is the proof finished?

Answer

Proof.

Per contra, we shall show in all following cases that the firm k ($k \in \{1, 2\}$ to be specified) would increase its profits by charging a price $p_k \neq p_k^*$.

First case: $\min\{p_1^*, p_2^*\} < c$. Takes $k = \arg\min_{i \in N}\{p_i^*\}$.

So firm k makes strictly negative profits.

A profitable deviation is to charge a higher price $p_k = c > p_k^*$. □

Proof.

Second case: $\min\{p_1^*, p_2^*\} > c$. Takes $k = \arg\max_{i \in N}\{p_i^*\}$.

A profitable deviation for firm k is to charge a lower price that is slightly below the competitor's one $p_k = p_j - \varepsilon$, $j \neq k$, $\varepsilon > 0$.

For ε small enough, the new price p_k is still higher than c so the resulting profit is strictly positive. □

Proof.

Third case: $\min\{p_1^*, p_2^*\} = c$. Then $\max\{p_1^*, p_2^*\} > c$. Takes $k = \arg\min_{i \in N}\{p_i^*\}$.

Firm k has a profitable deviation to charge a higher price that is slightly below the competitor's one $p_k = p_j - \varepsilon$, $j \neq k$, $\varepsilon > 0$ and small enough. □

Question

What happens in the asymmetric case where firm 1 has lower marginal cost $c_1 < c_2$?

Proposition

When $c_1 < c_2$:

- firm 2 makes no profit; and

- firm 1 charges price $p = c_2$ and makes a profit of $(c_2 - c_1) D(c_2)$

(as long as $c_2 \leq p^m(c_1) \in \arg\max_p (p - c_1) D(p)$; otherwise firm 1 charges its monopoly price $p^m(c_1)$).

Intuition

Firm 1 charges an ε below c_2 to make sure it has the whole market.

Remark (1)

In fact, there are equilibria where firm 1 charges c_2 (not an ε-below).

These rely on firm 2 randomizing uniformly over $[c_2, c_2 + \eta]$, for small enough $\eta > 0$. See, Blume (2003).

Remark (2)

Beyond this existence result, we "almost" have uniqueness:

In every Nash equilibrium in which firms use undominated strategies, the low-cost firm 1 serves the entire market at a price equal c_2.

See Kartik (2011).

- If there are n firms, each with a constant marginal cost satisfying $c_1 = c_2 = ... = c_{n-1} < c_n$ then $p^* = c_1$ and consumers are distributed among firm 1 to $n - 1$.

Bertrand Paradox

Conclusion

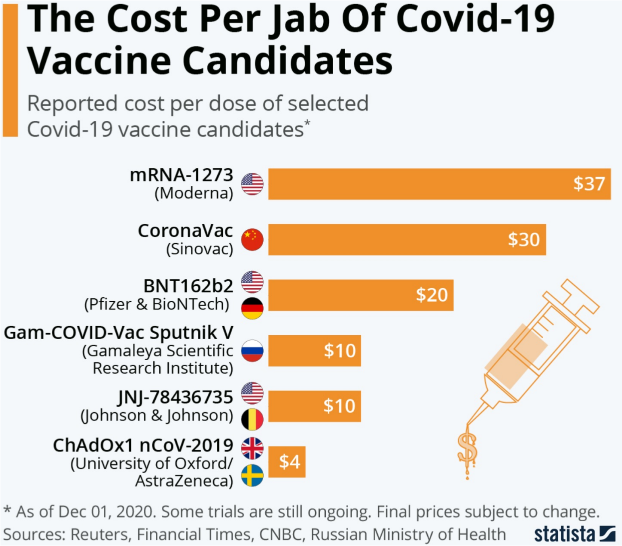
- When competing in prices, two firms (having the same marginal costs) is enough to replicate the pure and perfect competition.
- We have seen:
  - $c_1 = c_2 \Longrightarrow p^* = c_1$ and $\pi_1 = \pi_2 = 0$ (p.p.c.)
  - $c_1 < c_2 \Longrightarrow p^* = c_2$ and $\pi_1 > 0 = \pi_2$ (non p.p.c.)
  - $c_1 = c_2 < c_3 \Longrightarrow p^* = c_1$ and $\pi_1 = \pi_2 = \pi_3 = 0$ (p.p.c.).

Bertrand Paradox
Extension

Question

Is the Bertrand Paradox robust when introducing capacity constraints?

Answer


The Cost Per Jab Of Covid-19 Vaccine Candidates
Reported cost per dose of selected Covid-19 vaccine candidates*
mRNA-1273 (Moderna)
$37
CoronaVac (Sinovac)
$30
BNT162b2 (Pfizer & BioNTech)
$20
Gam-COVID-Vac Sputnik V (Gamaleya Scientific Research Institute)
$10
JNJ-78436735 (Johnson & Johnson)
$10
ChAdOx1 nCoV-2019 (University of Oxford/ AstraZeneca)
$4
* As of Dec 01, 2020. Some trials are still ongoing. Final prices subject to change.
Sources: Reuters, Financial Times, CNBC, Russian Ministry of Health
statista

- Assume that firm 1 has a production capacity smaller than $D(c)$.

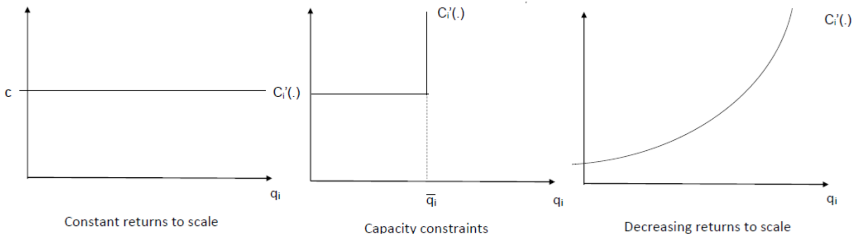
Question

Is $(p_1^*, p_2^*) = (c, c)$ still an equilibrium price system?

Answer

No, because if firm 2 increases its price slightly, it has a residual non-zero demand (since firm 1 cannot satisfy $D(c)$). So, firm 2 makes positive profits.

- The form of the residual-demand depends on which consumers are served by the low-price firm 1.
- Let us consider some decreasing returns to scale.
  - $C_i(q_i)$ is increasing and convex: $C_i' > 0$ and $C_i'' < 0$.
  - This is a generalization of capacity constraints (see Figure 4.4)



c
C'i(.)
qi
Constant returns to scale
C'i(.)
q̄i
qi
Capacity constraints
C'i(.)
qi
Decreasing returns to scale

Figure 4.4

- At a given price p, a firm is not willing to supply more than its competitive supply $S_i(p) \in \arg\max_q \pi(p,q) = \arg\max\{pq - C_i(q)\}$ which is defined by
$$p = C_i'(S_i(p))$$
- Assume that firm 1 has a capacity constraint, i.e., $S_1(p) < D(p)$ and $p_1 < p_2$.
    - So firm 2 faces some residual demand.

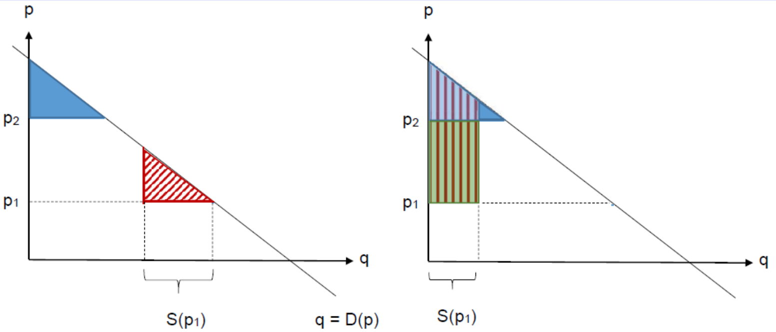
Question

If we want to maximize the consumers surplus which consumers shall we serve?

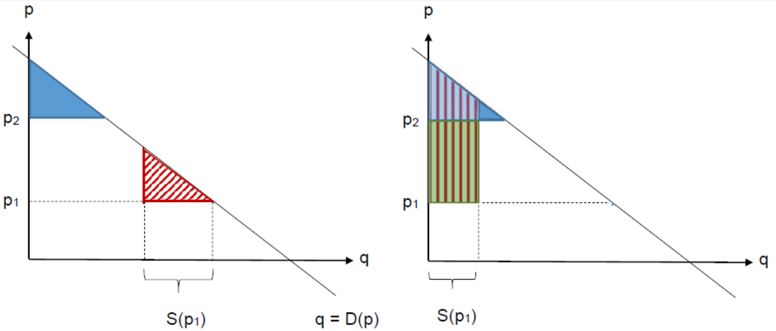
Answer

Proof.

[(Sketch)]



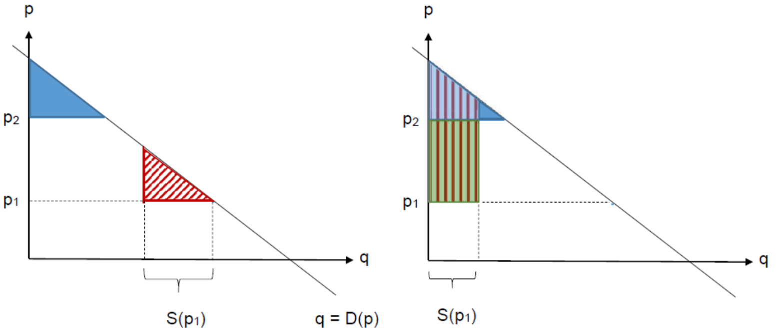
Proof.



On the LHS (resp. RHS) of Figure 4.5, the two areas (red and blue) depicts the total consumer surplus when serving the least (resp. most) eager agents (...) □

Proof.

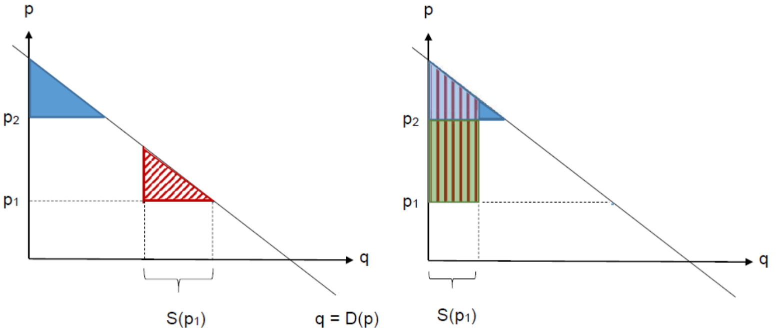
[(Sketch)]



In all cases, we have the red area because in all cases consumers with a valuation for the good higher than price p_2 will be served (...) □

Proof.

[(Sketch)]



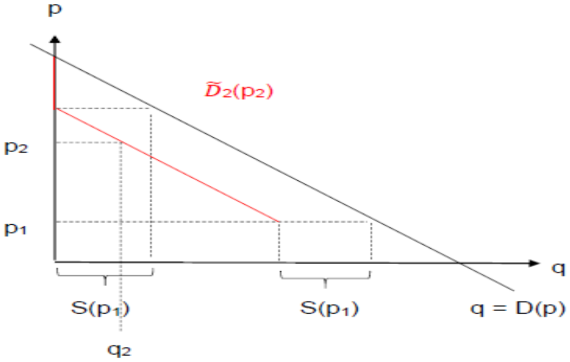
In the second case, we have in addition the green area defined by the rectangular $(p_2 - p_1)S(p_1)$. □

- This rationing is called the *efficient-rationing* rule because it maximizes consumers' surplus, even though it does not maximize trade!
  - If $D(p_2) < S(p_1)$, when serving the most eager consumers firm 2 will not sell anything.
  - While when serving the least eager, firm 1 will sell $S(p_1)$ and firm 2 will sell $D(p_2)$.
  - But, as we see with the green area the surplus is higher when serving the most eager consumers.
- Note that it would be obtained if the consumers were able to costlessly resell the good to each other.

- Concert offers a classical example where consumers with highest willingess to pay are served first.
  - consumers with highest willingess to pay are 1^{st} in queue; and
  - secondary market where consumers with low willingess to pay resell to consumers with higher willingess to pay.

- The efficient-rationing rule defines a residual function for firm 2:

$$\tilde{D}_2(p_2) = \begin{cases} D(p_2) - S(p_1) & \text{if} \quad D(p_2) > S(p_1) \\ 0 & \text{otherwise} \end{cases}.$$



- The *Proportional* or *Randomized-rationing* rule provide all consumers with the same probability of being rationed.
  - The probability of not being able to buy from firm 1 is:
  - $$\frac{D(p_1) - S(p_1)}{D(p_1)}$$
  - Hence, the residual demand facing firm 2 is:

$$\tilde{D}_2(p_2) = D(p_2)\left(\frac{D(p_1) - S(p_1)}{D(p_1)}\right).$$

Question

How to draw it?

Answer

$\tilde{D}_2(\cdot)$ is linear since $D(p_2)$ is linear in p_2 and for a fixed p_1, $\left(\frac{D(p_1)-S(p_1)}{D(p_1)}\right)$ is a constant.

Then we only need to know two points.

1) $D(p_2) = 0 \Longrightarrow \tilde{D}_2(p_2) = 0$

2) $\tilde{D}_2(p_1) = D(p_1) - S(p_1)$.

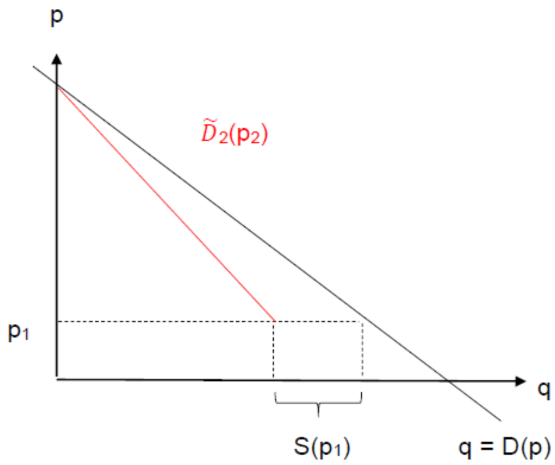


Figure 4.7

Question

Which rule firm 2 prefers? Said differently, under which rule firm 2's residual demand is higher at each price?

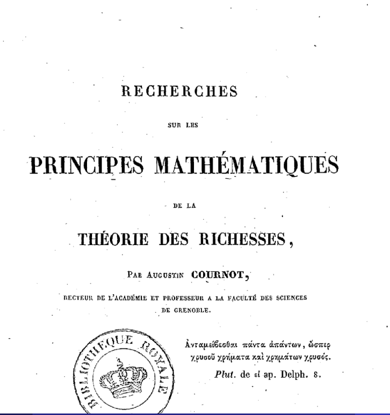
Answer

- We can see that graphically by comparing Figures 4.6 and 4.7.
  - And also analytically:

$$\begin{aligned} p_1 &< p_2 \Rightarrow D(p_1) > D(p_2) \Rightarrow -D(p_1)S(p_1) < -D(p_2)S(p_1) \\ &\Rightarrow D(p_1)D(p_2) - D(p_1)S(p_1) < D(p_1)D(p_2) - D(p_2)S(p_1) \\ &\Rightarrow D(p_2)\left(\frac{D(p_1)-S(p_1)}{D(p_1)}\right) > D(p_2) - S(p_1). \end{aligned}$$

Antoine Augustin Cournot (1801-1877)

- Cournot first outlined his theory of competition in 1838.
  - *Recherches sur les Principes Mathematiques de la Theorie des Richesses*
  - Contains explicit and mathematically precise models.


RECHERCHES

SUR LES

PRINCIPES MATHÉMATIQUES

DE LA

THÉORIE DES RICHESSES,

PAR AUGUSTIN COURNOT,

RECTEUR DE L'ACADÉMIE ET PROFESSEUR A LA FACULTÉ DES SCIENCES DE GRENOBLE.

Ἀνταμείβεσθαι πάντα ἁπάντων, ὥσπερ χρυσοῦ χρήματα καὶ χρημάτων χρυσός.

Plut. de εἰ ap. Delph. 8.

- Cournot described the competition with a market for spring water dominated by two suppliers (a duopoly).
  - He constructed profit functions for each firm
  - He then used partial differentiation to construct a function representing a firm's best response for given output levels of the other firm(s) in the market.
  - He showed that a stable equilibrium occurs where these functions intersect (i.e. the simultaneous solution of the best response functions of each firm).
  - In equilibrium, each firm's expectations of how other firms will act are shown to be correct; when all is revealed, no firm wants to change its output decision.
- This idea of stability was later taken up and built upon as a description of Nash equilibria, of which Cournot equilibria are a subset.
  - Cournot equilibrium (1838) is a Nash equilibrium (1950).

- $N = \{1, 2, ..., n\}$: n firms who:
  - produce a homogeneous product;
  - do not cooperate, i.e. there is no collusion;
  - have market power, i.e. each firm's output decision affects the good's price;
  - compete in quantities, and choose quantities simultaneously;
    - ⋆ E.g., oil extraction (if OPEC was not a cartel), agricultural products (sugar, cocoa, ...)
  - are rational and act strategically
    - ⋆ They seek to maximize profit given their competitors' decisions.

- Each firm $i \in N$:
  - has a production cost $C_i(q_i)$.
  - uses its production level $q_i \in \mathbb{R}$ as a strategy.
  - takes the quantity set by its competitors as a given, evaluates its residual demand, and then behaves as a monopoly.

- Total output $Q = \sum_{i=1}^{n} q_i$.
  - We denote $Q_{-i} := Q - q_i = \sum_{j=1, j \neq i}^{n} q_j$.
- Price adjusts to clear the market: $p = P(Q)$.
- Firm i's profit:

$$\pi_i (q_i, Q_{-i}) = q_i P(q_i + Q_{-i}) - C_i(q_i)$$

Definition

A profile $(q_1^*, q_2^*, ..., q_n^*)$ is a **Cournot equilibrium** if for all $i \in N$, we have

$$q_i^* \in \arg\max_{q_i} \pi_i \left(q_i, Q_{-i}^*\right)$$

with $Q_{-i}^* := \sum_{j=1, j \neq i}^{n} q_j^*$.

- At Cournot equilibrium, each firm maximizes its profit given the quantity chosen by the other firms.
  - So, Cournot equilibrium (1838) is a Nash equilibrium (1950).
  - It is also called a **(pure-strategy) Cournot-Nash equilibrium**.

- F.O.C.

$$\begin{aligned}\frac{\partial \pi_i (q_i, Q_{-i})}{\partial q_i} &= 0 \Longleftrightarrow \frac{\partial}{\partial q_i} \left(q_i P (q_i + Q_{-i}) - C_i (q_i) \right) = 0 \\ &\Longleftrightarrow \left[P (q_i + Q_{-i}) - C_i' (q_i) \right] + \left[q_i P' (q_i + Q_{-i}) \right] = 0\end{aligned}$$

  - The first bracket denotes the profitability of an extra unit of output
    - I.e., difference between price and marginal cost.
  - The second bracket denotes the profitability of inframarginal units
    - I.e., extra unit creates a decrease in price P', which affects the q_i units already produced.

- For a competitive firm $P'(\cdot) = 0$ because the firm is too small to affect the market price.
  - So, FOC writes as
$$P(q_i + Q_{-i}) = C_i'(q_i)$$
  - The firm prices at marginal cost.
- For a monopoly, $q_i = Q$ and $Q_{-i} = 0$
  - So, FOC writes as
$$P(Q) + P'(Q)Q = C_i'(Q)$$
  - The monopoly chooses a price such that the marginal revenue (LHS) equals the marginal cost (RHS).

- F.O.C.

$$P\left(q_i + Q_{-i}\right) - C_i'\left(q_i\right) + q_i P'\left(q_i + Q_{-i}\right) = 0$$

- The FOC illustrates the negative externality between the firms:
  - when choosing its output, firm i takes into account the adverse effect of the market price on its own output
    - ⋆ I.e., by considering $q_i P'\left(Q\right)$
  - rather than the effect on aggregate output
    - ⋆ I.e., by considering $QP'\left(Q\right)$.

- Hence each firm will tend to choose an output that exceeds the optimal output from the industry point of view (since $Q_{-i}P'(Q) < 0$).
- Thus the market price will be lower than the monopoly price.
- Also, the aggregate profit will be lower than the monopoly profit.

- FOC can be rewritten as the Lerner index (1934) which describes the firm i's market power:

$$L_i := \frac{P - C_i'(q_i)}{P}$$

with $L_i \in [0, 1]$ (higher index implies greater market power; $L_i = 0$ means no market power at all).

- By introducing the price-elasticity of demand facing firm *i*:

$$\varepsilon(p) := \frac{dD}{dp}\frac{p}{D} = p\frac{D'(p)}{D(p)}$$

which has the interpretation that p increasing by 1% yields the quantity demanded increases by $\varepsilon\%$.

  - Note that economists often refer to price-elasticity of demand as a positive value (i.e., in absolute value terms: $\varepsilon(p) := -p\frac{D'(p)}{D(p)}$) with the interpretation that p increasing by 1% yields the quantity demanded *decreases* by $\varepsilon\%$.

- It is sometimes useful to rewrite Lerner index as a function of the individual market share $\frac{q_i}{Q}$ and elasticity:

$$L_i = \frac{P - C_i'(q_i)}{P} = -\frac{q_i}{Q}\frac{1}{\varepsilon(P)}$$

The second equality comes from our previous F.O.C. according to which $P - C_i'(q_i) + q_i P' = 0$, so
$\frac{P - C_i'(q_i)}{P} = -\frac{q_i P'}{P} = -\frac{q_i\left(\frac{dP}{dD}\right)}{P} = \frac{q_i\left(\frac{P}{-\varepsilon(P)D}\right)}{P} = q_i\frac{1}{-\varepsilon(P)Q}$.

- $L_i > 0$ since $D'(p) < 0 \Longrightarrow \varepsilon(p) < 0$.
  - So firms sells at a price exceeding marginal cost.
  - Thus, the Cournot equilibrium is not socially efficient.

- $D(p) = 1 - p$
- Constant return to scale: $C_i(q_i) = c_i q_i$
- Each firm chooses q_i that solves

$$\max_{q_i} \left(\pi_i\left(q_i, Q_{-i}\right)\right)$$

with

$$\pi_i\left(q_i, Q_{-i}\right) = \left(1 - q_i - Q_{-i}\right) q_i - c_i q_i$$

- Assuming $q_i > 0$ for all $i \in N$, FOC is

$$1 - 2q_i - Q_{-i} = c_i$$
$$\iff \quad 1 - q_i - Q = c_i$$

- Summing over all q_i yields:

$$n - Q - nQ = \sum_{i=1}^{n} c_i$$

- Thus the Cournot equilibrium aggregate industry output and market price are

$$Q = \frac{n - \sum_{i=1}^{n} c_i}{n+1} \quad \text{and} \quad p = 1 - Q = \frac{1 + \sum_{i=1}^{n} c_i}{n+1}$$

- Also, we find

$$\begin{aligned} q_i &= 1 - Q - c_i = p - c_i = \frac{1 + \sum_{i=1}^{n} c_i}{n+1} - c_i \\ &= \frac{1 + \sum_{j \neq i} c_j - nc_i}{n+1} \end{aligned}$$

- So a firm's output decreases with its marginal cost and increases with its competitors' marginal costs.

- From the previous section, when $n = 2$, we get:
  - the firm j's reaction curve write as:

$$q_j\left(q_i\right) = \frac{1 - c_j - q_i}{2}$$

  - the Cournot equilibrium firm j's output writes as:

$$\begin{aligned} q_j^* &= q_j^*\left(q_i^*\left(q_j\right)\right) = \frac{1 - c_j}{2} - \left(\frac{1 - c_i - q_j}{2}\right) \\ &= \frac{1 + c_i - 2c_j}{3} \end{aligned}$$

- We can depict the reaction curves in the (q_1, q_2) space:

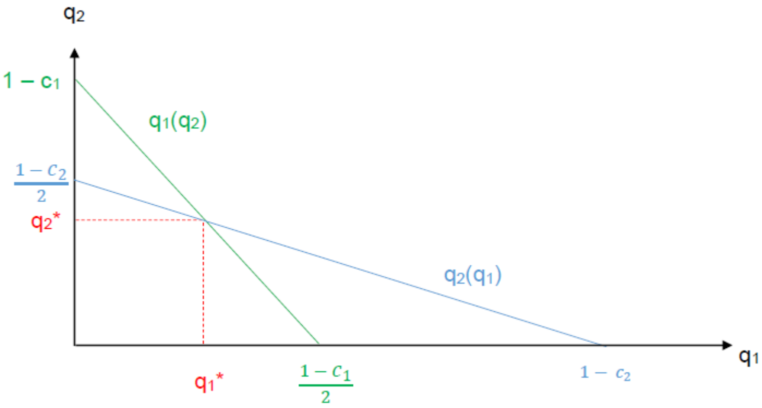


Figure 4.10

Question

What would be the effect of an increase in firm 1's marginal cost?

Answer

It would have the effect of decrease firm 1's output and increase firm 2's output.

- Indeed,...

- Indeed, if $c_1 \rightarrow c_1' > c_1$ we get $q_1'^* < q_1^*$ and $q_2'^* > q_2^*$.

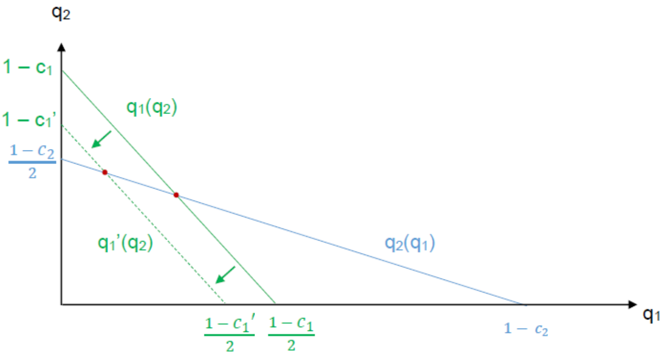


Figure 4.11

- $c_i = c$ for all $i \in N$
- We then obtain a symmetric equilibrium (i.e., $q_i^* = q$, for all $i \in N$) given by:

$$\frac{p-c}{p} = \frac{1}{n}\frac{1}{\varepsilon(p)} \quad and \quad Q = nq = n\frac{1-c}{n+1}$$

  and

$$q = \frac{1-c}{n+1} \quad ; \quad p = 1 - nq = c + \frac{1-c}{n+1} \quad and \quad \pi_i = \pi = \frac{(1-c)^2}{(n+1)^2}$$

- Varying the number of firms:
  - $n = 1$: monopoly situation;
  - $n \to +\infty$: $\lim_{n\to+\infty} Q = 1 - c$ and $\lim_{n\to+\infty} p = c$, competitive solution.

- F. H. Hahn (1962): “The Stability of the Cournot Oligopoly Solution”, *The Review of Economic Studies,* Vol. 29, No. 4, pp. 329-331

Definition

Firm i's **reaction function** is defined by $R_i : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ with

$$R_i\left(Q_{-i}\right) := \arg\max_{q_i} \pi_i\left(q_i, Q_{-i}\right).$$

- So,

$$R_i\left(Q_{-i}\right) = \arg\max_{q_i} \left\{q_i P\left(q_i + Q_{-i}\right) - C_i\left(q_i\right)\right\}.$$

- Observe that the assumption $R_i : \mathbb{R} \longmapsto \mathbb{R}$, means that firm i only focuses on the total quantity $Q_{-i} \in \mathbb{R}$.
  - Firm i could rather take into account on which competitor produces what quantity.
    - We then would have $R_i : \mathbb{R}^{n-1} \longmapsto \mathbb{R}$with $R_i \left((q_j)_{j \neq i} \right)$.

- We can now rewrite the definition of Cournot equilibrium wrt reaction functions.

Definition

A profile $(q_1, q_2, ..., q_n)$ is a (pure-strategy) **Cournot-Nash equilibrium** if for all $i \in N$, we have

$$q_i = R_i (Q - q_i)$$

with $Q := \sum_{i=1}^{n} q_i$.

- Said differently, such an equilibrium is obtained by finding an aggregate output such that

$$Q = \sum_{i=1}^{n} q_i(Q)$$

that is, a fixed point of the function

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

where $q_i(Q)$ solves $P(Q) + q_i P'(Q) - C_i(q_i) = 0$ or is equal to zero if this equation has no positive solution.

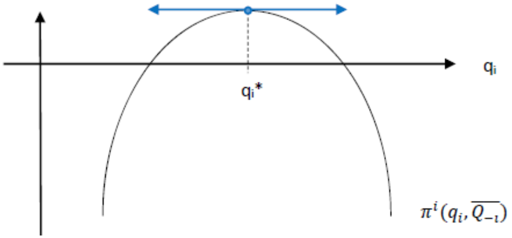
- If $\pi_i(q_i, Q_{-i})$ is strictly concave then the reaction function $R_i(\cdot)$ is:
  - (I.e., if $\frac{\partial^2 \pi_i(q_i, Q_{-i})}{\partial q_i^2} = 2P'(q_i + Q_{-i}) + q_i P''(q_i + Q_{-i}) - C_i''(q_i) < 0$.)
  - continuous, single-valued and defined by the FOC.
  - 

Figure 4.12

- The FOC writes as

$$\frac{\partial \pi_i \left(q_i, Q_{-i}\right)}{\partial q_i} = 0$$

which rewrites as

$$P\left(q_i + Q_{-i}\right) + q_i P'\left(q_i + Q_{-i}\right) - C_i'\left(q_i\right) = 0$$

that is, since $R_i\left(Q_{-i}\right) := \arg\max_{q_i} \pi_i\left(q_i, Q_{-i}\right)$,

$$P\left(R_i\left(Q_{-i}\right) + Q_{-i}\right) + R_i\left(Q_{-i}\right) P'\left(R_i\left(Q_{-i}\right) + Q_{-i}\right) - C_i'\left(R_i\left(Q_{-i}\right)\right) = 0$$

- $R_i\left(Q_{-i}\right)$ is decreasing if

$$\frac{\partial}{\partial Q_{-i}}\left(\frac{\partial \pi_i\left(q_i, Q_{-i}\right)}{\partial q_i}\right) < 0$$

that is

$$\frac{\partial^2 \pi_i\left(q_i, Q_{-i}\right)}{\partial q_i \partial Q_{-i}} < 0$$

Definition

Hahn conditions are:

$$\frac{\partial^2 \pi_i \left(q_i, Q_{-i}\right)}{\partial q_i \partial Q_{-i}} < 0$$

and

$$P' \left(q_i + Q_{-i}\right) - C_i'' \left(q_i\right) < 0$$

Proposition

Under Hahn conditions the Cournot equilibrium exists and is unique.

Proof.

First Hahn condition write as

$$\frac{\partial^2 \pi_i (q_i, Q_{-i})}{\partial q_i \partial Q_{-i}} < 0$$

that is

$$P' (q_i + Q_{-i}) + q_i P'' (q_i + Q_{-i}) < 0$$

Summing this, to the second Hahn condition

$$P' (q_i + Q_{-i}) - C_i'' (q_i) < 0$$

we obtain

$$2P' (q_i + Q_{-i}) + q_i P'' (q_i + Q_{-i}) - C_i'' (q_i) < 0$$

Proof.

which means that

$$\frac{\partial^2 \pi_i (q_i, Q_{-i})}{\partial q_i^2} < 0$$

So $R_i(\cdot)$ is continuous, single-valued and decreasing.
So is

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

The Brouwer theorem asserts that a continuous function from a compact set into itself admits at least one fixed point. □

Proof.

Here, compactness is easily obtained from

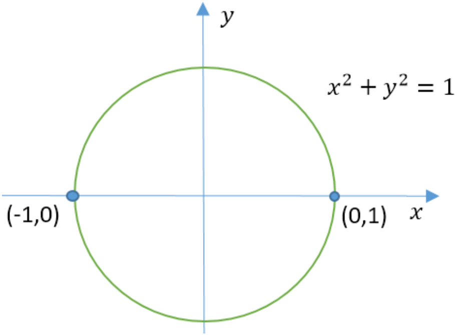
$$+\infty > q_i(0) \geq q_i(Q) \geq 0$$

where:
- the first inequality comes from the fact that each firm would produce a finite quantity if it were a monopoly;
- the second inequality comes from the fact that $q_i(\cdot)$ is decreasing; and
- the third one from the definition of q_i.

The equilibrium then exists. □

- To establish uniqueness, we need to apply the *Implicit Function Theorem.*

- A function assigns a *single* value in the range for every value in the domain.
  - It is really covenient because we generally know how to compute the derivative and the integral of it.
- The problem is that some mathematical objects are not a function.
  - E.g., a circle defined by
$$x^2 + y^2 = 1$$
is not a function even though it describe a relationship between x and y.



- The idea of the *Implicit Function Theorem* is to use the fact that almost every point can locally (i.e., in a neighborhood) be described as a function.
  - E.g., only two points in our circle cannot: $(-1, 0)$ and $(1, 0)$.

- The *Implicit Function Theorem* provides conditions under which a relationship (not necessarily a function) of the form $F(x,y)=0$ can be rewritten as a function $y=f(x)$ locally (in a small neighborhood of a point).
  - E.g., our circle can be described by the relationship $F(x,y)=x^2+y^2-1$, which in turns for positive y take the form of $y=\sqrt{1-x^2}$, and for negatives y takes the form of $y=-\sqrt{1-x^2}$.
  - The Theorem is called *implicit* because it does not provide us with the explicit formulae of the function $f(\cdot)$, but rather just ensures its existence.

Theorem (*Implicit Function Theorem*)

Let $F(x,y) \in C^1$ *in a neighborhood of* (x_0, y_0) *such that:*

$$F(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0$$

Then there exists a neighborhood of (x_0, y_0) *in which there is an implicit function* $y = f(x)$ *such that:*

(i). $f(x_0) = y_0$;

(ii). $F(x, f(x)) = 0$ *for every* x *in the neighborhood; and*

(iii). $f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ *in the neighborhood.*

- The Theorem can be rewritten as:
  - Let
$$\begin{array}{rccc} F: & E\times F & \longmapsto & G \\ & (x,y) & \longmapsto & F(x,y) \end{array}$$
there is a function $f: E' \longmapsto F'$, with $E' \subset E$ and $F' \subset F$ such that
$$\begin{array}{rccc} F: & E'\times F' & \longmapsto & G \\ & (x,y) & \longmapsto & F(x,f(x)) \end{array}$$
  - $F(x,f(x)) = 0$ implies that $0 = F'_x(\cdot,\cdot) + \frac{\partial F(x,f(x))}{\partial f(x)} f'(x)$ so $f'(x) = -\frac{F'_x(\cdot,\cdot)}{F'_y(\cdot,\cdot)}$.

Proof.

By considering the function

$$F\left(x,f\left(x\right)\right)=\frac{\partial\pi_i\left(Q_{-i},R_i\left(Q_{-i}\right)\right)}{\partial R_i}$$

we obtain

$$\begin{aligned} R_i'\left(Q_{-i}\right) &= -\frac{\frac{\partial^2\pi_i(Q_{-i},R_i(Q_{-i}))}{\partial Q_{-i}\partial R_i}}{\frac{\partial^2\pi_i(Q_{-i},R_i(Q_{-i}))}{\partial R_i^2}} \\ &= -\frac{1}{1+\frac{P'(q_i+Q_{-i})-C_i''(q_i)}{P'(q_i+Q_{-i})+q_iP''(q_i+Q_{-i})}}\in\left(-1,0\right) \end{aligned}$$

□

Proof.

Now from $q_i(Q) = R_i(Q - q_i(Q))$ we obtain

$$q_i'(Q) = R_i'(Q - q_i(Q)) \times \left(1 - q_i'(Q)\right)$$

so

$$q_i'(Q) = \frac{R_i'(Q - q_i(Q))}{1 + R_i'(Q - q_i(Q))}$$

which is negative since $R_i'(Q - q_i(Q)) \in (-1, 0)$.
Thus,

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

is strictly decreasing in Q and the equilibrium is unique. □

- Inexistence due to discountinuity

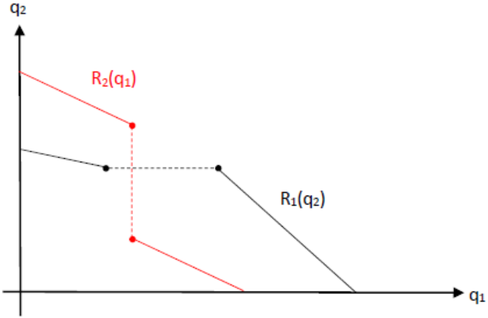


Figure 4.14

- Multiplicity

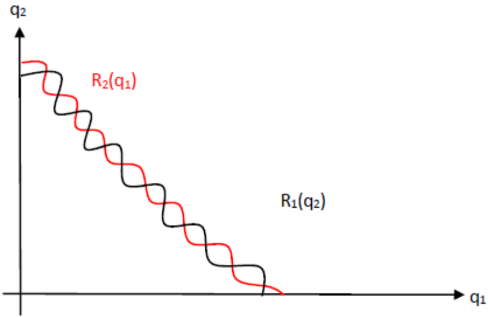


Figure 4.15

- To be parallel R_i and R_j must be such that $R_i' = \frac{1}{R_j'}$ that is excluded by Hahn conditions (since $R_i' \in (-1, 0)$).

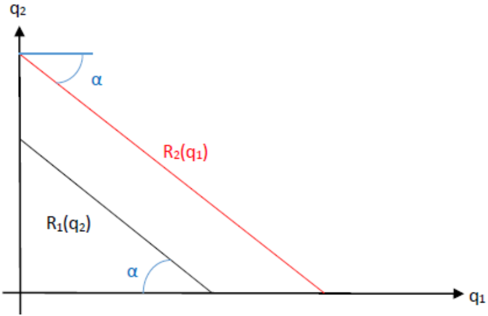


Figure 4.16
Here, $R_2' = \alpha$ and $R_1' = \frac{1}{\alpha}$

- Regular Cournot equilibrium
  - $R_j^{-1}(0) > q_i^m$ $(= R_i(0))$: firm i's output that induces firm j to produce nothing exceeds firm i's monopoly output.

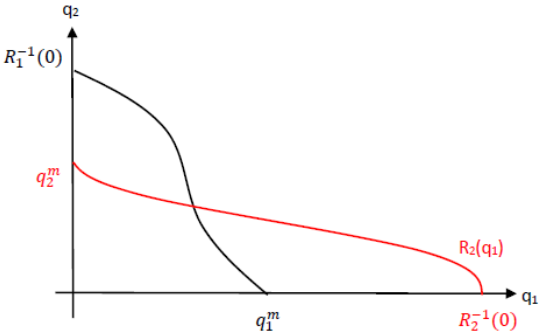


Figure 4.17

- Cournot outcome (output and price) is somewhere in between monopoly and perfect competition.
  - Aggregate output (resp. price) is greater (resp. lower) with Cournot duopoly than monopoly.
  - Aggregate output (resp. price) is lower (resp. greater) with Cournot duopoly than perfect competition.
- Firms have an incentive to form a cartel, effectively turning the Cournot model into a Monopoly.
  - Cartels are usually illegal, so firms might instead tacitly collude using self-imposing strategies to reduce output which, ceteris paribus will raise the price and thus increase profits for all firms involved.
  - We shall study it later...

- Cournot vs Bertrand:
  - Bertrand. More realistic assumption: firms compete in price (not quantity).
  - Cournot. More realistic prediction: two firms are not enough to push prices down to marginal cost level and then restore pure and perfect competition.
  - As the number of firms increases towards infinity, the Cournot model gives the same result as in Bertrand model
    - ⋆ The market price is pushed to marginal cost level.

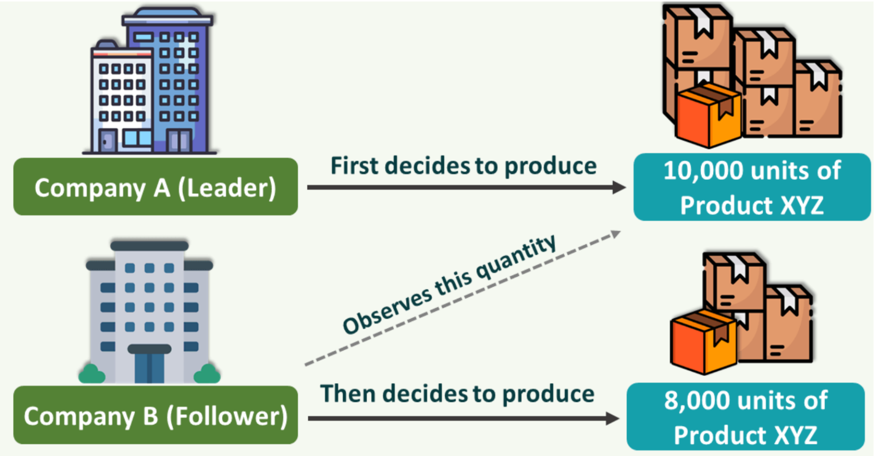
- Neither model (Bertrand or Cournot) is necessarily better.
  - The accuracy of the predictions of each model will vary from industry to industry, depending on the closeness of each model to the industry situation.
  - If capacity and output can be easily changed, Bertrand is a better model of duopoly competition.
  - If output and capacity are difficult to adjust, then Cournot is generally a better model.
- We shall see later how to recast Cournot and Bertrand altogether as a two-stage model.

Stackelberg: sequential moves
Introduction

Heinrich Freiherr von Stackelberg (1905-1946)
Published *Market Structure and Equilibrium* (*Marktform und Gleichgewicht*) in 1934

Stackelberg: sequential moves

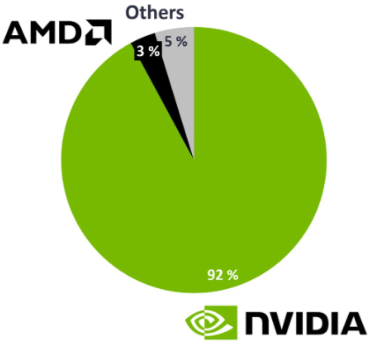
Introduction



THE
GENERATIVE
AI
MARKET

- Data Centers' GPU
  - Data center GPUs (Graphics Processing Unit) serve as the foundational building blocks for Generative AI, providing the computational power required to train powerful machine learning models.
  - GPU are well-suited for AI applications because of their parallel processing capabilities and efficiency in handling large amounts of data simultaneously.
- The GPUs market reached $49 billion in 2023. A booming increase from 2022 (+182%), mostly driven by one company alone: NVIDIA (92%).

Introduction: Generative AI Example



- Stackelberg-like market:
  - NVIDIA is the clear leader of the market based on factors of quality and flexibility.
  - The followers mainly compete for the most favorable performance-to-price ratio.

Introduction: Generative AI Example

Stock prices on 20 June 2024

- We consider a duopoly where firms move in sequence.
  - Firm 1: leader
  - Firm 2: follower
  - As in Cournot, competition is on quantity.
- Firms may engage in Stackelberg competition if one:
  - has some sort of advantage enabling it to move first;
  - is the incumbent monopoly of the industry and the follower is a new entrant;
  - is holding excess capacity; or
  - has commitment power.

Stackelberg: sequential moves
Result

Question

Is there any advantage for moving in the first stage rather than the second?

- We solve the game by backward induction.

- Second period subgames
  - Firm 2 chooses q_2 to maximize its profit given firm 1's quantity.
    - ⋆ Identical to the problem of firms in the Cournot market structure.
    - ⋆ Best response function of firm 2: $R_2(q_1)$.
- First period game

$$\max_{q_1} q_1 P(q_1 + R_2(q_1)) - C_1(q_1)$$

- FOC writes as

$$P\left(q_1 + R_2\left(q_1\right)\right) - C_1'\left(q_1\right) + q_1 P'\left(q_1 + R_2\left(q_1\right)\right)\left(1 + R_2'\left(q_1\right)\right) = 0$$

- With respect to Cournot, we see that there is a new term in the FOC: $\left(1 + R_2'\left(q_1\right)\right)$.
    - By adding this term, LHS becomes smaller than zero because $P'\left(\cdot\right) < 0$, and by Hahn conditions $R_2'\left(\cdot\right) \in \left(-1, 0\right)$.
    - So, we must decrease $R_2'\left(\cdot\right)$ to come back to zero.
    - Since $R_2'\left(\cdot\right) < 0$, we then must increase q_1.
    - Hence, $q_1^{Stackelberg} > q_1^{Cournot}$.

Stackelberg: sequential moves

Result

Question

Since the leader takes into account the follower's reaction function in his optimization program, why the leader does not behave as in the Cournot equilibrium?

Answer

Stackelberg: sequential moves

Result

Example

$1 \backslash 2$	L	R
U	2,0	0,1
D	1,1	0,0

$BR^1(L) = \{U\}$; $BR^1(R) = \{U, D\}$; $BR^2(U) = \{R\}$; $BR^2(D) = \{L\}$.
So, the pure-strategy Nash equilibrium is (U, R).

When Player 1 moves at 1^{st}, the game becomes:

$1 \backslash 2$	$BR^2(\cdot)$
U	0,1
D	1,1

and (D, L) is the Stackelberg outcome.

Stackelberg: sequential moves
Result

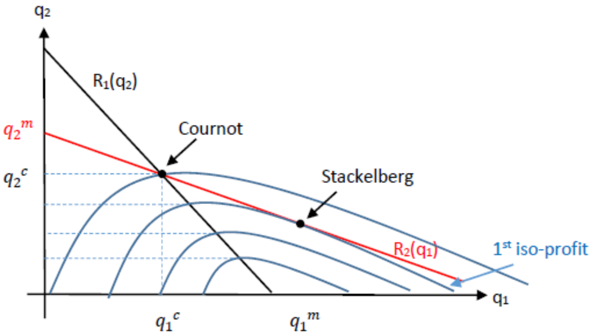


Figure 4.18

- Under Hahn conditions, $R_2'(\cdot) \in (-1, 0)$ so $q_1^{Cournot} + q_2^{Cournot} < q_1^{Stackelberg} + q_2^{Stackelberg}$.

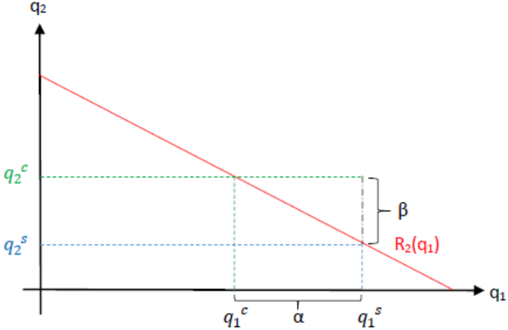


Figure 4.19

- Stackelberg outcome (output and price) is somewhere in between monopoly and perfect competition.
    - Aggregate output (resp. price) is greater (resp. lower) with Stackelberg than monopoly.
    - Aggregate output (resp. price) is lower (resp. greater) with Stackelberg than perfect competition.
- Stackelberg outcome is somewhere in between Cournot and Bertrand.
    - Aggregate output (resp. price) is greater (resp. lower) with Stackelberg than Cournot.
    - Aggregate output (resp. price) is lower (resp. greater) with Stackelberg than Bertrand.
    - Consumer surplus is greater (resp. lower) with Stackelberg than Cournot (resp. Bertrand).

Outline

Question

How to recast Cournot and Bertrand altogether as a two-stage model?

David M. Kreps and Jose A. Scheinkman. “Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes”, *The Bell Journal of Economics*, Vol. 14, No. 2 (Autumn, 1983), pp. 326-337

Capacity and price game
Model

- $N = \{1, 2\}$
- 2 stages game:
  - Fimrs choose production capacities (i chooses $\bar{q}_i$)
  - Fimrs choose price (i chooses p_i)
- Demand function is concave
  - $P'(\cdot) < 0$ and $P''(\cdot) \leq 0$
- $C_i(q_i) = cq_i$ and $C_i(\bar{q}_i) = c_i\bar{q}_i$.
- Efficient rationing rule
  - $p_1 < p_2$ with $\bar{q}_1 < D(p_1)$ implies that the residual for firm 2 writes as

$$D(p_2, \bar{q}_1) = \max\{D(p_2) - \bar{q}_1, 0\}$$

Capacity and price game
Results

- To solve this two stages game, we proceed by backward induction.
  - We start by fixing the capacity constraint (1st period choices) to solve the resulting price game (2nd period choices).
  - Once the second stage best responses are characterized, we characterize the first stage best responses.

- Suppose firm i has a rigid capacity constraint $\bar{q}_i$

Proposition

Firms price at marginal cost if and only if $D(c) \leq \min\{\bar{q}_1, \bar{q}_2\}$.

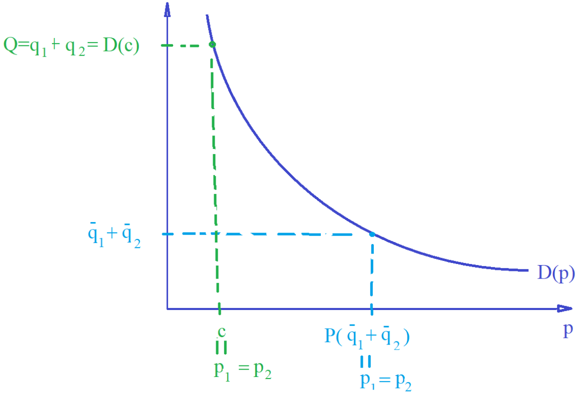
Proof.

"$\Longleftarrow$". This is Bertrand's Proposition

"$\Longrightarrow$". Assume, per contra, $p_1 = p_2 = c$ and $D(c) > \min\{\bar{q}_1, \bar{q}_2\} = \bar{q}_1$. Then firm 2 can set $p_2 = c + \varepsilon$ and face a positive residual demand, a contradiction. □

Proposition

In a pure-strategy equilibrium, firms sell up to capacity, i.e., $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$.



Proof.

First, let us show that $p_1 = p_2$.

Suppose, *per contra*, $p_1 < p_2$.

If $D(p_1) > \bar{q}_1$ then firm 1 woud be better-off by raising its price, a contradiction.

If $D(p_1) \leq \bar{q}_1$ then firm 1 supplies all the demand at price p_1.

– if $p_1 \leq c$ then firm 1 can increase its price above c and realize positive profit, a contradiction.

– if $p_1 > c$ firm 2 can realize positive (instead of zero) profit by charging $p_2 = p_1 - \varepsilon$, with $\varepsilon \in (0, c - p_1)$, a contradiction. □

Proof.

Second, let us show that $p = P(\bar{q}_1 + \bar{q}_2)$.

– If $D(p) > \bar{q}_1 + \bar{q}_2$ then both firms ration their consumers. Each firm could increase its price and still sell its capacity.

– If $D(p) < \bar{q}_1 + \bar{q}_2$ then the price is too high and one firm at least cannot sell its capacity. By charging $p - \varepsilon$ this firm would sell its capacity. □

Let $R_i(.)$ denote firm i's Cournot best response to the other firm's capacity.

Proposition

There is a pure-strategy equilibrium in prices only if $\bar{q}_i \leq R_i(\bar{q}_j)$ for all i.

Proof.

Assume, per contra, $\bar{q}_i > R_i(\bar{q}_j)$ and a pure-strategy equilibrium in prices does exist.

By the previous Proposition, the pure-strategy equilibrium in prices satisfies $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$.

From $\bar{q}_i > R_i(\bar{q}_j)$ we have $P(\bar{q}_i + \bar{q}_j) < P(R_i(\bar{q}_j) + \bar{q}_j)$. So $p_i < P(\bar{q}_j + R_i(\bar{q}_j))$. (...) □

Proof.

If firm i is capacity constrained then it can raises its price slightly and make profit $(p_i + \varepsilon)\,\bar{q}_i > p_i \bar{q}_i$.

If not, firm j must be capacity constrained (otherwise they would set lower prices). That is, $q_j = \bar{q}_j$. So by definition of the reaction function $R_i(\cdot)$, firm i's best responds by charging $p_i = P(R_i(\bar{q}_j) + \bar{q}_j)$, a contradiction. □

- Conclusion
  - According to the first Proposition, for high capacities (*i.e.*, $D(c) \leq \min\{\bar{q}_1, \bar{q}_2\}$) we have the Bertrand equilibrium outcome (*i.e.*, prices equals marginal cost).
  - According to the two other Propositions, for low capacities (*i.e.*, $\bar{q}_i \leq R_i(\bar{q}_j)$ for all i) we have the Cournot equilibrium outcome (*i.e.*, $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$).
  - For intermediate capacities we have no pure-strategy equilibrium. Further, the highest capacity firm makes a profit equal to its Stackelberg follower profit.
    - I.e., $\pi^F(\bar{q}_j) = R_i(\bar{q}_j)\left(P(R_i(\bar{q}_j) + \bar{q}_j) - c\right)$
    - See Kreps and Scheinkman (1983) for the complete proof, and Tirole (1988, MIT) for sketch).

- Let us now add a prior and simultaneous choice of capacities.
- Each firm has capacity $\bar{q}_i$ with cost $c_i\bar{q}_i$ and then decides to produce q_i with cost cq_i.
- Adding the capacity cost to the first period will not change the ssecond period reasonning because this cost is sunk.
- 2nd stage firm i's profit becomes:

$$\max_{q_i \leq \bar{q}_i} q_i \left(P\left(q_i + \bar{q}_j\right) - c\right) - c_i\bar{q}_i$$

- Given that firms sell up to capacity, the capacity choice $\bar{q}_i$ solves

$$\max_{\bar{q}_i} \bar{q}_i \left(P\left(\bar{q}_i + \bar{q}_j\right) - c - c_i\right)$$

Proposition

The Cournot outcome $(\bar{q}_1 = q^*, \bar{q}_2 = q^*)$ *where* q^* *maximizes* $q\left(P\left(q + q^*\right) - c - c_i\right)$ *is an equilibrium.*

Proof.

Suppose that firm i plays q^*. Firm j, if it plays $q \leq R\left(q^*\right)$ (where $R\left(\cdot\right)$ still denotes the 2^{nd} stage reaction function), by definition of q^*, gets

$$q\left(P\left(q + q^*\right) - c - c_i\right) \leq q^*\left(P\left(2q^*\right) - c - c_i\right)$$

If firm j plays $q > R\left(q^*\right)$ then it gets the Stackelberg follower profit.

$$\begin{aligned}\pi^F\left(q^*\right) &= R\left(q^*\right)\left(P\left(q^* + R\left(q^*\right)\right) - c - c_i\right) \\ &\leq q^*\left(P\left(2q^*\right) - c - c_i\right).\end{aligned}$$

□

- The Cournot equlibrium is the equilibrium in the 1^{st}-stage capacity game and the 2^{nd}-stage price is equal to $P\left(2q^*\right)$.
- The capacity game is a Cournot game with total producing costs $\left(c + c_i\right)\bar{q}_i$.
- To prove uniqueness in the choice of capacities require more work (see, Kreps and Scheinkman, 1983).

- Kreps and Scheinkman (1983) show that the difference between Cournot and Bertrand competition is more than just the strategy space, but that timing of decisions is also relevant.
  - To illustrate this, they study and solve a Bertrand like duopoly model of competition where timing of decision is inverted.
    - ⋆ Capacity decision is made simultaneously and before price decision (as opposed to Bertrand models where the choice of capacity and price is interpreted as being simultaneous), and the low priced firm may not serve all the demand at her price (as it is in the Bertrand approach) due to capacity constraints.
    - ⋆ In a two stage game where firms first set simultaneously capacity and then engage in simultaneous price competition with demand rationed following the efficient rationing rule, the unique Subgame Perfect Nash Equilibrium (SPNE) has as outcome the Cournot quantities and prices.

- Davidson and Deneckere (RAND, 1986) argue that the Kreps and Scheinkman result depends strongly on the chosen rationing rule.
- Madden (ET, 1998) shows, in a slightly different framework, that for uniformly elastic demands the Kreps and Scheinkman result holds, even if proportional rationing.

Conclusion

- In the standard Monopoly model we obtain the same result regardless of the choice variable of the Monopolist (price or quantity)
- This no longer holds for the Oligopoly models.
  - The equilibrium outcome depends crucially on the strategic variable.
  - Bertrand model: price.
  - Cournot model: quantity.
  - Kreps and Scheinkman model: quantity and price.
  - Flath (2012) finds out that on 70 Japanese manufacturing industries, 5 are Cournot-like, 35 are Bertrand-like, and 30 are hybrid-like.

Conclusion

- In Oligopoly models, the order of moves also plays a role.
  - Bertrand and Cournot models: simultaneous moves.
  - Stackelberg model: as Cournot (quantity competition) but sequential moves.
    - ⋆ The leader selects the pair (own quantity, rival's response quantity) that maximizes its profits.
  - Kreps and Scheinkman model: sequential stages of simultaneous moves.

Question

Which competition model among price (Bertrand), quantity (Cournot), leader-follower (Stackelberg) and capacity-then-price fits better with the following industries:

- movie theater;
- VOD (Video on Demand) ;
- pharmaceutical ;
- air transport;
- car rental;
- concert;
- electric automobile ?

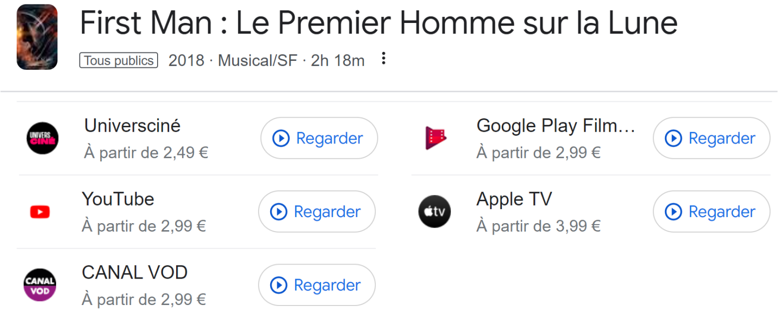
Answer: Movie theaters

Answer

.

- They compete by offering a variety of films and showtimes to attract audiences.
- The focus is on filling seats rather than setting prices very high or low.

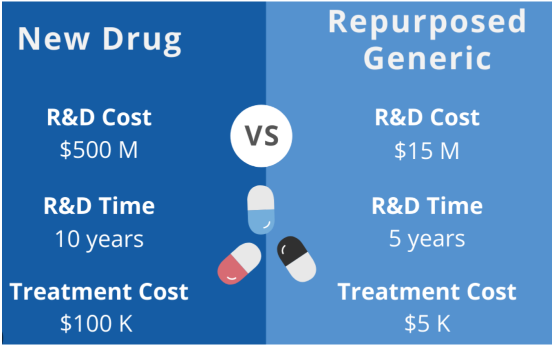
Answer: VOD



Answer

- Platforms compete by setting subscription fees or pay-per-view prices.
- The emphasis is on attracting subscribers by offering competitive pricing and content libraries.

Answer: Pharmaceutical



Answer

- Pharmaceutical companies often engage in R&D with one large company (e.g. Pfizer, Merck, Johnson & Johnson) introducing a new drug (leader), and others following with similar products (generics).

Answer: Air Transport

Answer

- Airlines often adjust the number of available seats (capacity) based on demand and then set prices accordingly.
- This is known as "yield management".

Answer: Car Rental

Answer

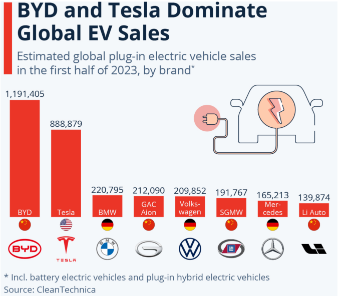
- While capacity (availability of rental cars) is a consideration, price tends to be a more direct and visible competitive factor.
- Companies compete by offering competitive rental rates, discounts, and promotions.

Answer: Concert

Answer

- The success of a concert is often measured by the number of tickets sold, and artists and promoters compete by attracting larger audiences.
- Pricing may play a role, but the focus is on filling the venue.

Answer: Electric automobile



Answer

- Large electric automobile companies (e.g. Tesla, Byd) act as leaders and set the trends for the industry.
- They influence prices and control production levels, which affects the profitability of the followers (Li-auto, Nio...).

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