Introduction to Derivative Instruments

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Slides on book: John C. Hull, "Options, Futures, and Other Derivatives", Pearson ed.

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Chapter 9

Chapter 9: Properties of Stock Options

Outline

Introduction

Motivation

- We shall use a number of different arbitrage arguments to explore the relationships between European option prices, American option prices, and the underlying stock price.
  - The most important of these relationships is put-call parity, which is a relationship between the price of a European call option, the price of a European put option, and the underlying stock price.
- We shall also examine whether American options should be exercised early.
  - We shall see that in most cases it is not optimal to exercise an American call option on a non-dividend-paying stock prior to the option's expiration.
  - When there are dividends, it can be optimal to exercise either calls or puts early.

Chapter 9: Properties of Stock Options

Outline

Factors Affecting Option Prices
Notation

- Call option price:
  - c: European
  - C: American
- Put option price:
  - p: European
  - P: American
- Stock price:
  - S_0: today
  - S_T: at maturity T

Factors Affecting Option Prices
Notation

- K: Strike price
- σ: Volatility of stock price
  - Roughly speaking, the volatility of a stock price is a measure of how uncertain we are about future stock price movements.
- D: PV of dividends paid during life of option
- r: Risk-free rate for maturity T with continuous compounding.

Factors Affecting Option Prices
Effect of Variables on Option Pricing

Variable	c	p	C	P
S_0	$+$	$-$	$+$	$-$
K	$-$	$+$	$-$	$+$
T	?	?	$+$	$+$

- S_0: the higher the stock price today, the higher (resp. lower) the call (resp. put) option price.
- K: Call (resp. put) options become less (resp. more) valuable as the strike price increase.
- T: American options become more valuable as the time to expiration increases.
  - European options usually become more valuable as the time to expiration increases, but this is not always the case.
    - If a very large dividend is expected then an european call option that expires before the dividend is paid is worth more than an european call option that expires after the dividend is paid.

Factors Affecting Option Prices
Effect of Variables on Option Pricing

Variable	c	p	C	P
σ	$+$	$+$	$+$	$+$
D	$-$	$+$	$-$	$+$

- σ: As volatility increases, the chance that the stock will do very well or very poorly increases.
  - For the owner of a stock, these two outcomes tend to offset each other.
  - However, the owner of an option benefits from one particular direction price change but has limited downside risk in the event of a change in the opposite direction, because the most the owner can lose is the price of the option.
  - The values of both calls and puts therefore increase as volatility increases.
- D: Dividends have the effect of reducing the stock price on the ex-dividend date.
  - This is bad news for the value of call options and good news for the value of put options.

Factors Affecting Option Prices
American vs European Options

- An American option is worth at least as much as the corresponding European option

$$C \geq c \qquad and \qquad P \geq p.$$

Chapter 9: Properties of Stock Options
Outline

Upper and Lower Bounds for Option Prices
Upper Bounds

- An American or European call option gives the holder the right to buy one share of a stock for a certain price.
  - No matter what happens, the option can never be worth more than the stock.
  - Hence, the stock price is an upper bound to the option price:

$$c \leq S_0 \qquad and \qquad C \leq S_0.$$

- If these relationships were not true, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option.

Upper and Lower Bounds for Option Prices
Upper Bounds

- An American put option gives the holder the right to sell one share of a stock for K.
  - No matter how low the stock price becomes, the option can never be worth more than K.
  - Hence,

$$P \leq K.$$

- For European options, we know that at maturity the option cannot be worth more than K.
  - It follows that it cannot be worth more than the present value of K today:

$$p \leq Ke^{-rT}.$$

- If this were not true, an arbitrageur could make a riskless profit by writing the option and investing the proceeds of the sale at the risk-free interest rate.

Upper and Lower Bounds for Option Prices
Lower Bound for Calls on Non-Dividend-Paying Stocks

Proposition

A lower bound for the price of a European call option on a non-dividend-paying stock is

$$\max(S_0 - Ke^{-rT}, 0).$$

Proof.

Consider the following two portfolios:
Portfolio A: one European call option plus a zero-coupon bond that provides a payoff of K at time T.
Portfolio B: one share of the stock. □

Upper and Lower Bounds for Option Prices
Lower Bound for Calls on Non-Dividend-Paying Stocks

Proof.

In portfolio A, at time T we have:

The zero-coupon bond is worth K.

If $S_T > K$, the call option is exercised.

The portfolio A is then worth $(S_T - K) + K = S_T$.

If $S_T < K$, the call option expires worthless

The portfolio A is then worth K.

Hence, at time T portfolio A is worth $\max(S_T, K)$.

Portfolio B is worth S_T at time T. □

Upper and Lower Bounds for Option Prices
Lower Bound for Calls on Non-Dividend-Paying Stocks

Proof.

Hence, portfolio A is always worth as much as, and can be worth more than, portfolio B at the option's maturity.
It follows that in the absence of arbitrage opportunities this must also be true today.
The zero-coupon bond is worth Ke^{-rT} today. Hence,

$$c + Ke^{-rT} \geq S_0 \quad \Longleftrightarrow \quad c \geq S_0 - Ke^{-rT}.$$

Because the worst that can happen to a call option is that it expires worthless, its value cannot be negative.
This means that $c \geq 0$ and therefore

$$c \geq \max(S_0 - Ke^{-rT}, 0).$$

□

Upper and Lower Bounds for Option Prices
Lower Bound for European Puts on Non-Dividend-Paying Stocks

Proposition

For a European put option on a non-dividend-paying stock, a lower bound for the price is

$$\max(Ke^{-rT} - S_0, 0)$$

Proof.

Consider the following two portfolios:
Portfolio C: one European put option plus one share.
Portfolio D: a zero-coupon bond paying off K at time T.
If $S_T < K$, then the option in portfolio C is exercised at option maturity and the portfolio becomes worth K.
If $S_T > K$, then the put option expires worthless and the portfolio is worth S_T at this time.
Hence, portfolio C is worth $max\{S_T, K\}$ in time T. □

Proof.

Portfolio D is worth K in time T.
Hence, portfolio C is always worth as much as, and can sometimes be worth more than, portfolio D in time T.
It follows that in the absence of arbitrage opportunities portfolio C must be worth at least as much as portfolio D today.
Hence,

$$p + S_0 \geq Ke^{-rT} \quad \Longleftrightarrow \quad p \geq Ke^{-rT} - S_0.$$

Because $p \geq 0$ we have

$$p \geq \max(Ke^{-rT} - S_0, 0).$$

□

Proposition (Put-Call Parity)

The value of a European call with a exercise price K and exercise date T can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa, according to the following formula

$$c + Ke^{-rT} = p + S_0.$$

Proof.

Consider the two A and C used in the proof of the previous proposition.
We have already shown that they are both worth $\max\{S_T, K\}$ at time T.
Since the portfolios have identical values at time T, they must have identical values today. □

Proof.

The components of portfolio A are worth c and Ke^{-rT} today.
The components of portfolio C are worth p and S_0 today.
Hence,

$$c + Ke^{-rT} = p + S_0.$$

□

Question

What would be the arbitrage opportunity if $c + Ke^{-rT} < p + S_0$?

Solution

Upper and Lower Bounds for Option Prices
Put-Call Parity

Solution

Upper and Lower Bounds for Option Prices
Put-Call Parity

Question

What would be the arbitrage opportunity if $c + Ke^{-rT} > p + S_0$?

Solution

Upper and Lower Bounds for Option Prices
Put-Call Parity

Solution

Upper and Lower Bounds for Option Prices
Put-Call Parity

Proposition (Put-Call Parity)

When there are no dividends, an American option satisfies

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}.$$

Proof.

Homework. (Hint: For the first part of the relationship consider (a) a portfolio consisting of a European call plus an amount of cash equal to K and (b) a portfolio consisting of an American put option plus one share.) □

Upper and Lower Bounds for Option Prices
Early Exercise of Calls on a Non-Dividend Paying Stock

There are two reasons an American call on a non-dividend-paying stock should not be exercised early.

- One relates to the insurance that it provides.
  - A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price.
  - Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes.
- The other reason concerns the time value of money.
  - From the perspective of the option holder, the later the strike price is paid out the better.
  - Indeed, we have $C \geq c \geq S_0 - Ke^{-rT} \geq S_0 - K$.
    - This means that C is always greater than the option's intrinsic value prior to maturity.

Upper and Lower Bounds for Option Prices
Upper and lower bounds

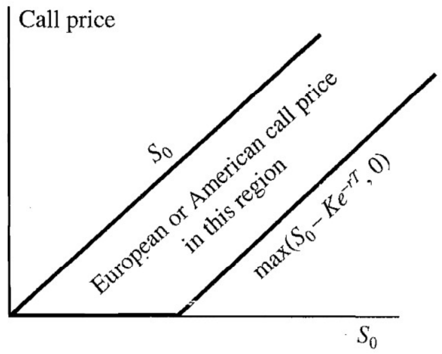
- Because American call options are never exercised early when there are no dividends, they are equivalent to European call options, so that $C = c$.
- It follows that upper and lower bounds are given by

$$\max(S_0 - Ke^{-rT}, 0) \leq c = C \leq S_0.$$

Upper and Lower Bounds for Option Prices
Upper and lower bounds

- We then have the following figure

Figure 10.3 Bounds for European and American call options when there are no dividends.



Upper and Lower Bounds for Option Prices
Early Exercise on Puts on a Non-Dividend Paying Stock

- It can be optimal to exercise an American put option on a non-dividend-paying stock early.
  - To illustrate, consider an extreme situation.
    - Suppose that the strike price is \$10 and the stock price is virtually zero.
    - By exercising immediately, an investor makes an immediate gain of \$10.
    - If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10, because negative stock prices are impossible.
    - Furthermore, receiving \$10 now is preferable to receiving \$10 in the future. It follows that the option should be exercised immediately.
  - Therefore at any given time during its life, a put option should always be exercised early if it is sufficiently deep in the money.

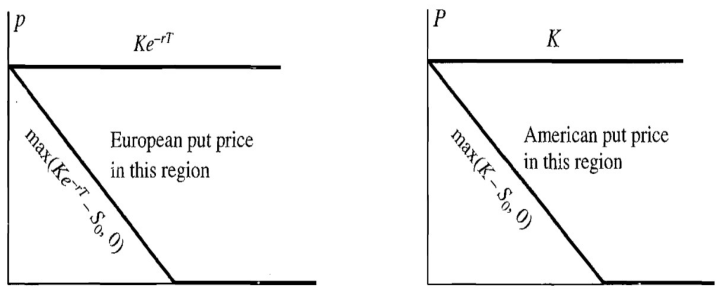
Upper and Lower Bounds for Option Prices
Upper and lower bonds on Puts on a Non-Dividend Paying Stock

- For European options, when there is no dividend we already know that
$$\max(Ke^{-rT} - S_0, 0) \leq p \leq Ke^{-rT}.$$
- For an American put option on a non-dividend-paying stock, we have
$$\max(K - S_0, 0) \leq P$$
because the option can be exercised at any time.
- American put option on a non-dividend-paying stock then satisfies
$$\max(K - S_0, 0) \leq P \leq K$$

Upper and Lower Bounds for Option Prices
Upper and lower bonds on Puts on a Non-Dividend Paying Stock

- We then have the following figure

Figure 10.5 Bounds for European and American put options when there are no dividends.



Exercise (6)

The price of an American call on a non-dividend-paying stock is \$4.

The stock price is \$31, the strike price is \$30, and the expiration date is in three months.

The risk-free interest rate is 8%.

Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

Solution (6)

Upper and Lower Bounds for Option Prices
The Impact of Dividends on Lower Bounds to Option Prices

- Let D denotes the present value of the dividends during the life of the option.
- We have
$$\max(S_0 - D - Ke^{-rT}, 0) \leq c$$
and
$$\max(D + Ke^{-rT} - S_0, 0) \leq p$$

- We also have

$$c + D + Ke^{-rT} = p + S_0$$

and

$$S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT}.$$

Exercise (5)

The price of a European call that expires in six months and has a strike price of \$30 is \$2.

The underlying stock price is \$29, and a dividend of \$0.50 is expected in two months and again in five months.

The term structure is flat, with all risk-free interest rates being 10%.

What is the price of a European put option that expires in six months and has a strike price of \$30?

Solution (5)

Chapter 9: Properties of Stock Options
Outline

Summary

- We have examine five factors affecting the value of a stock option:
  - the current stock price;
  - the strike price;
  - the expiration date;
  - the stock price volatility; and
  - the dividends expected during the life of the option.

Summary

- The value of a call generally:
  - increases as the current stock price, the time to expiration, and the volatility increase; and
  - decreases as the strike price and expected dividends increase.
- The value of a put generally:
  - increases as the strike price, the time to expiration, the volatility, and the expected dividends increase; and
  - decreases as the current stock price increase.
- It is possible to reach some conclusions about the value of stock options without making any assumptions about the volatility of stock prices.
  - For example, the price of a call option on a stock must always be worth less than the price of the stock itself.
  - Similarly, the price of a put option on a stock must always be worth less than the option's strike price.

Summary

- A European call option on a non-dividend-paying stock must be worth more than

$$\max(S_0 - Ke^{-rT}, 0)$$

where S_0 is the stock price, K is the strike price, r is the risk-free interest rate, and T is the time to expiration.
- A European put option on a non-dividend-paying stock must be worth more than

$$\max(Ke^{-rT} - S_0, 0)$$

- When dividends with present value D will be paid, the lower bound for a European call option becomes

$$\max(S_0 - D - Ke^{-rT}, 0)$$

and the lower bound for a European put option becomes

$$\max(e^{-rT} + D - S_0, 0)$$

Summary

- Put-call parity is a relationship between the price, c, of a European call option on a stock and the price, p, of a European put option on a stock.
  - For a non-dividend-paying stock, it is

$$c + Ke^{-rT} = p + S_0$$

- For a dividend-paying stock, the put-call parity relationship is

$$c + D + Ke^{-rT} = p + S_0$$

- Put-call parity does not hold for American options. However, it is possible to use arbitrage arguments to obtain upper and lower bounds for the difference between the price of an American call and the price of an American put.