Introduction to Derivative Instruments

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Jérôme MATHIS

`www.jeromemathis.fr/Derivatives`
password: `272-Derivatives`

Slides on book: John C. Hull, “Options, Futures, and Other Derivatives”, Pearson ed.

LEDa

Chapter 5

Chapter 5: Interest Rates
Outline

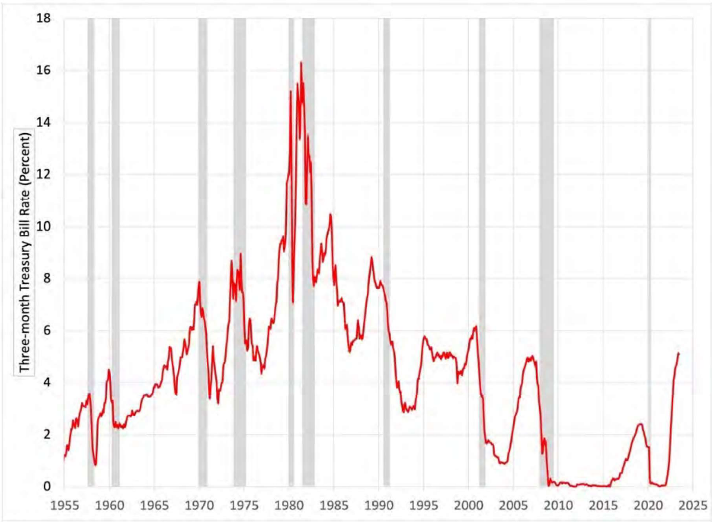
Motivation and Types of Rates
Motivation

- Interest rates are a factor in the valuation of virtually all derivatives.
- An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender.
- For any given currency, many different types of interest rates are regularly quoted.
  - These include mortgage rates, deposit rates, prime borrowing rates, and so on.
- The interest rate applicable in a situation depends on the credit risk.
  - This is the risk that there will be a default by the borrower of funds, so that the interest and principal are not paid to the lender as promised.
  - The higher the credit risk, the higher the interest rate that is promised by the borrower.

Motivation and Types of Rates
Types of Rates

- **Treasury rates** are the rates an investor earns on Treasury bills and Treasury bonds.
  - These are the instruments used by a government to borrow in its own currency.
  - It is usually assumed that there is no chance that a government will default on an obligation denominated in its own currency.
    - Treasury rates are therefore usually assumed totally risk-free rates.

Motivation and Types of Rates
Types of Rates



Note: Gray shading denotes recessions. Source: Federal Reserve Economic Data

Motivation and Types of Rates
Types of Rates

- **LIBOR** is the rate of interest at which a bank is prepared to deposit money with another bank.
  - LIBOR is short for the *London Interbank Offered Rate*.
  - It is quoted in all major currencies for maturities up to 12 months:
    - ⋆ E.g., 3-month LIBOR is the rate at which 3-month deposits are offered.
  - A deposit with a bank can be regarded as a loan to that bank.
    - ⋆ A bank must therefore satisfy certain creditworthiness criteria in order to be able to receive deposits from another bank at LIBOR.
    - ⋆ Typically it must have a AA credit rating.

Motivation and Types of Rates
Types of Rates

- **LIBID** is is the rate at which banks will accept deposits from other banks.
  - LIBID is short for the *London Interbank Bid Rate*.
- At any specified time, there is a small spread between LIBID and LIBOR rates (with LIBOR higher than LIBID).
  - The rates themselves are determined by active trading between banks and adjust so that the supply of funds in the interbank market equals the demand for funds in that market.
- **Repurchase agreement, or Repo,** is a contract where:
  - an investment dealer who owns securities agrees to sell them to another company now and buy them back later at a slightly higher price; and
  - the other company is providing a loan to the investment dealer.
- **Repo rate** is the rate of a repurchase agreement.
  - It is calculated from the difference between the price at which the securities are sold and the price at which they are repurchased.

Motivation and Types of Rates
Types of Rates

- **Risk-Free Rate**
  - The short-term risk-free rate traditionally used by derivatives practitioners is not Treasury rate.
  - The Treasury rate is considered to be artificially low for a number of reasons
    - ⋆ 1. Treasury bills and Treasury bonds must be purchased by financial institutions to fulfill a variety of regulatory requirements.
    - ⋆ This increases demand for these Treasury instruments driving the price up and the yield down.
    - ⋆ 2. The amount of capital a bank is required to hold to support an investment in Treasury bills and bonds is substantially smaller than the capital required to support a similar investment in other instruments with very low risk.
    - ⋆ 3. In the United States, Treasury instruments are given a favorable tax treatment compared with most other fixed-income investments because they are not taxed at the state level.

Motivation and Types of Rates
Types of Rates

- **Risk-Free Rate (cont')**
  - Eurodollar futures and swaps are used to extend the LIBOR yield curve beyond one year
  - Following the credit crisis that started in 2007, many dealers switched to using overnight indexed swap rates instead of LIBOR as risk-free rates
    - Banks became very reluctant to lend to each other during the subprime crisis and LIBOR rates soared.

Motivation and Types of Rates
Types of Rates

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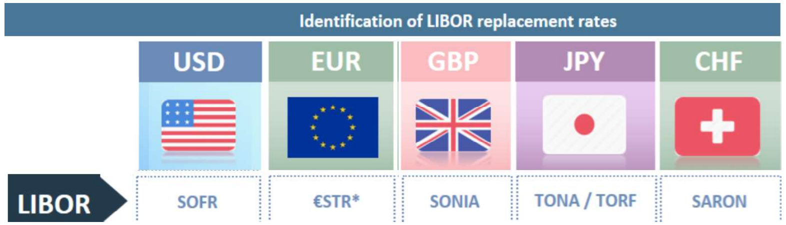
  - Following LIBOR scandal (2012), LIBOR Dollar has been progressively replaced by Secured Overnight Financing Rate (SOFR) from 2018 to 2023.
    - FED calculates SOFR based on Repo market.
  - Similarly, EONIA (Euro OverNight Index Average) has been progressively replaced by Euro Short-Term Rate (€STR said "Ester") from 2019 to 2022.
    - ECB calculates €STR based on overnight unsecured fixed rate deposit transactions over €1 million among the 50 largest banks in the euro area.



Chapter 5: Interest Rates
Outline

Measuring Interest Rates

- If the interest rate is measured with annual compounding, a bank's statement that the interest rate is 10% means that \$100 grows to

$$\$100 \times 1.1 = \$110$$

  at the end of 1 year.
- When the interest rate is measured with semiannual compounding, it means that 5% is earned every 6 months, with the interest being reinvested. In this case \$100 grows to

$$\$100 \times \left(1 + \frac{10\%}{2}\right)^2 = \$100 \times (1.05)^2 \simeq \$110.25$$

  at the end of 1 year.

Measuring Interest Rates

- When the interest rate is measured with quarterly compounding, the bank's statement means that 2.5% is earned every 3 months, with the interest being reinvested. The $100 then grows to

$$\$100 \times \left(1 + \frac{10\%}{4}\right)^4 = \$100\,(1.025)^4 \simeq \$110.38$$

at the end of 1 year.

Measuring Interest Rates
Generalization

- To generalize our results, suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded once per annum, the terminal value of the investment is

.

- If the rate is compounded m times per annum, the terminal value of the investment is

.

Definition

When $m = 1$, the rate is referred to as the **equivalent annual interest rate**.

Measuring Interest Rates

- Table 1 shows the effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	*Value of $100 at end of year ($)*
Annually ($m = 1$)	110.00
Semiannually ($m = 2$)	110.25
Quarterly ($m = 4$)	110.38
Monthly ($m = 12$)	110.47
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.52

Measuring Interest Rates
Continuous Compounding

Question

We observe that the amount in right column of Table 1 is increasing with the compounding frequency. Is there any upper bound on this amount?

Definition

The limit as the compounding frequency, m, tends to infinity is known as **continuous compounding**.

Solution

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Measuring Interest Rates
Continuous Compounding

Property

We have

Proof.

From the Taylor serie we know that

$$f(a+x) = \qquad + \ldots$$

$$= \qquad .$$

Measuring Interest Rates
Continuous Compounding

Proof.

So we obtain the following Maclaurin serie

$$\ln(1+x) = \qquad \text{for } |x| < 1.$$

So

$$\frac{1}{x}\ln(1+x) = \qquad + \ldots$$

For $x = \frac{R}{m}$ with $m > R$ we have $|\frac{R}{m}| < 1$ so

$$\frac{1}{R}\left(\ln\left(1+\frac{R}{m}\right)^m\right) = \frac{1}{\frac{R}{m}}\ln\left(1+\frac{R}{m}\right) = 1 - \frac{R}{2m} + \frac{R^2}{3m^2} - \frac{R^3}{4m^3} + \ldots$$

□

Measuring Interest Rates
Continuous Compounding

Proof.

That is

$$\ln\left(1+\frac{R}{m}\right)^m = R\left(1 - \frac{R}{2m} + \frac{R^2}{3m^2} - \frac{R^3}{4m^3} + \ldots\right)$$

$$= R - \frac{R^2}{2m} + \frac{R^3}{3m^2} - \frac{R^4}{4m^3} + \ldots$$

Hence

That is

and

□

Measuring Interest Rates
Continuous Compounding

- How to to convert a rate with a compounding frequency of m times per annum to a continuously compounded rate and vice versa?

Solution

Suppose that R_c is a rate of interest with continuous compounding and R_m is the equivalent rate with compounding m times per annum.

We have

That is

and

Measuring Interest Rates
Continuous Compounding

Question

Suppose that a lender quotes the interest rate on loans as 10% per annum with continuous compounding, and that interest is actually paid monthly.

What is the equivalent rate with monthly compounding?

What are the interest payments required each month on a $25,000 loan?

Measuring Interest Rates
Continuous Compounding

Solution

The equivalent rate with monthly compounding writes as

with $m =$, *and* $R_c =$.
That is

$$R_{12} =$$

This means that on a $25,000 *loan, interest payments of*

would be required each month.

Exercise (4)

A bank quotes you an interest rate of 14% per annum with quarterly compounding.

What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

Solution (4)

Chapter 5: Interest Rates
Outline

Zero Rates and bonds

Definition

A **zero rate** (or **spot rate**), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T.

Example

Suppose a 3-year zero rate with continuous compounding is quoted as 2% per annum. This means that \$100, if invested for 3 years, grows to

$$100e^{0.02\times 3} \simeq 106.18.$$

- Problem: Most of the interest rates we observe directly in the market are not pure zero rates.
  - Consider a 3-year government bond that provides a 2% coupon. The price of this bond does not by itself determine the 3-year Treasury zero rate because some of the return on the bond is realized in the form of coupons prior to the end of year 3.

Zero Rates and bonds
What is the difference between bills, notes and bonds?

- **Treasury bills** (T-Bills), **notes** and **bonds** are marketable securities the U.S. government sells in order to pay off maturing debt and to raise the cash needed to run the federal government.
- A **principal** (which is also known as **par value** or **face value**) is what is paid at the end of the security life (minus the final coupon, if any).
- **T-bills** are short-term obligations issued with a term of one year or less, and because they are sold at a discount from face value, they do not pay interest before maturity.
  - In other words, they are short-term zero coupon bonds.
  - The interest is the difference between the purchase price and the price paid either at maturity (face value) or the price of the bill if sold prior to maturity.

Zero Rates and bonds
What is the difference between bills, notes and bonds?

- **Treasury notes** and **bonds**, on the other hand, are securities that have a stated interest rate that is paid periodically (usually semi-annually) until maturity.
  - What makes notes and bonds different are the terms to maturity. Notes are issued in two-, three-, five- and 10-year terms. Conversely, bonds are long-term investments with terms of more than 10 years.
- In all this chapter we will use the generic word “bonds” to refer to T-bills, notes, and bonds.

Zero Rates and bonds
Bond Pricing

Question

What is the theoretical price of a 2-year Treasury bond with a principal of \$100 providing coupons at the rate of 6% per annum semiannually, when the zero rates are given by the following table?

Table 4.2 Treasury zero rates.

Maturity (years)	*Zero rate (%) (continuously compounded)*
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Solution

This bond will deliver:

- *\$3 in 6 months;*
- *\$3 in 12 months;*
- *\$3 in 18 months; and*
- *\$103 in 24 months.*

According to the table, these amounts have a present value of

$$\simeq 98.39$$

Definition

The **bond yield** is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond.

Question

Suppose that the theoretical price of the bond we have been considering, \$98.39, is also its market value.

What is the corresponding bind yield?

Solution

The bond yield (continuously compounded), denoted as y, is given by solving

to get $y = 0.0676$ *or* 6.76%.

- **Remark:** One way of solving nonlinear equations of the form $f(x) = 0$, such as the one of the previous solution (with $x = y$ and $f(x)$ is LHS minus RHS), is to use the Newton's method. We start with an estimate x_0 of the solution and produce successively better estimates x_1, x_2, ... using the formula $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ until a sufficiently accurate value is reached.

Definition

The **par yield** for a certain maturity is the coupon rate that causes the bond price to equal its par value.

Question

Suppose that the coupon on a 2-year bond in our example is c per annum (or $\frac{c}{2}$ per 6 months).

What is the corresponding 2-year par yield?

Zero Rates and bonds
Par Yield

Solution

Using the zero rates in Table 4.2, the value of the bond is equal to its par value of 100 when

so $c = 6.87$.

The 2-year par yield is therefore 6.87% *per annum (with semiannual compounding).*

Zero Rates and bonds
Par Yield

- In general if m is the number of coupon payments per year, d is the present value of \$1 received at maturity and A is the present value of an annuity of \$1 on each coupon date we have

$$c = \frac{(100 - 100d)\, m}{A}$$

- In our example, $m = 2$, $d = e^{-0.068\times 2}$, and $A = e^{-0.05\times 0.5} + e^{-0.058\times 1} + e^{-0.064\times 1.5} + e^{-0.068\times 2} \simeq 3.700$ so

$$c \simeq \frac{\left(100 - 100e^{-0.068\times 2}\right) 2}{3.7} \simeq 6.87\%$$

The formula confirms that the par yield is 6.87% per annum.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

- Since new three months, six months, and one year T-bills are traded publicly, we can look up their yields from database (internet, newspaper, etc.).
- It may be the case that there is no 18-month *zero-coupon* Treasury issue traded publicly at the moment.
- We can use 18-month *coupon-bearing* Treasury security to deduce it.
- The most popular approach is an iterative process called the **bootstrap method** which consists in
  - First, defining a set of yielding products (e.g., coupon-bearing bonds).
  - Second, deriving discount factors for all terms recursively, by forward substitution.
  - Doing so, we 'Bootstrap' the zero-coupon curve step-by-step.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Exercise

Deduce the Treasury zero rates of Table 4.2 from the following Table 4.3 that gives the prices of five bonds.

Table 4.3 Data for bootstrap method.

Bond principal (\$)	*Time to maturity* (*years*)	*Annual coupon** (\$)	*Bond price* (\$)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

* Half the stated coupon is assumed to be paid every 6 months.

- Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Solution (3-month bond)

The 3-month bond has the effect of turning an investment of 97.5 into 100 in 3 months. The continuously compounded 3-month rate R is therefore given by solving

It is 10.127% per annum.

Solution (6-month bond)

The 6-month continuously compounded rate is similarly given by solving

It is 10.469% per annum.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Solution (1-year bond)

Similarly, the 1-year rate with continuous compounding is given by solving

It is 10.536% per annum.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Solution (18-month bond)

The fourth bond lasts 1.5 years. The payments are as follows:

- *6 months: $4;*
- *1 year: $4;*
- *1.5 years: $104.*

From our earlier calculations, we know that

$$R_{0.5} = 0.10469$$

and

$$R_{1.0} = 0.10536$$

.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Solution (18-month bond)

So $R_{1.5}$ satisfies

that is $R_{1.5} \simeq 0.10681$.

This is the only zero rate that is consistent with the 6-month rate, 1-year rate, and the data in Table 4.3.

Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Solution (2-year bond)

The 2-year zero rate can be calculated similarly from the 6-month, 1-year, and 1.5-year zero rates, and the information on the last bond in Table 4.3.
If $R_{2.0}$ *is the 2-year zero rate, then*

This gives $R_{2.0} = 0.10808$, *or* 10.808%.

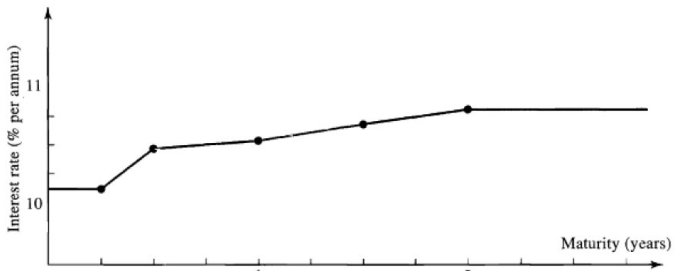
Zero Rates and bonds
Determining Treasury zero rates using Bootstrap Method

Definition

The zero curve is a chart showing the zero rate as a function of maturity.

- A common assumption is that the zero curve is linear between the points determined using the bootstrap method.

Figure 4.1 Zero rates given by the bootstrap method.



Chapter 5: Interest Rates
Outline

Forward Rates and Forward Rate Agreement
Forward Rates

Definition

The **forward rate** is the future zero rate implied by today's term structure of interest rates.

Example

The forward interest rate for year 2 is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year.

Forward Rates and Forward Rate Agreement
Forward Rates

- Consider Table 4.5 in which the second column gives the LIBOR zero rates.

Table 4.5 Calculation of forward LIBOR rates.

Year (n)	*Zero rate for an n-year investment (% per annum)*	*Forward rate for nth year (% per annum)*
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

- The forward interest rate for year 2 can be calculated from the 1-year zero interest rate of 3% per annum and the 2-year zero interest rate of 4% per annum.
- It is the rate of interest for year 2 that, when combined with 3% per annum for year 1, gives 4% overall for the 2 years.

Forward Rates and Forward Rate Agreement
Forward Rates

Question

According to the third column of Table 4.5 the forward interest rate for year 2 is 5% per annum.
Is it correct?

Forward Rates and Forward Rate Agreement
Forward Rates

Solution

Suppose that \$100 is invested. A rate of 3% for the first year and 5% for the second year gives

at the end of the second year.

A rate of 4% per annum for 2 years gives

- Remark: The result is only approximately true when the rates are not continuously compounded.

Forward Rates and Forward Rate Agreement
Forward Rates

- The forward rate for year 3 is the rate of interest that is implied by a 4% per annum 2-year zero rate and a 4.6% per annum 3-year zero rate.
- According to the third column of Table 4.5 it is 5.8% per annum.
  - The reason is that an investment for 2 years at 4% per annum combined with an investment for one year at 5.8% per annum gives an overall average return for the three years of 4.6% per annum.
- In general, if R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively, and R_F is the forward interest rate for the period of time between T_1 and T_2, then

$$e^{R_2 T_2} = e^{R_1 T_1} e^{R_F (T_2 - T_1)}, \text{ that is } R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

Forward Rates and Forward Rate Agreement
Forward Rates

- The previous formula can be written as

$$R_F =$$

- This shows that if the zero curve is upward sloping between T_1 and T_2, so that $R_2 > R_1$, then $R_F > R_2$.
  - I.e., the forward rate for a period of time ending at T_2 is greater than the T_2 zero rate.
- Similarly, if the zero curve is downward sloping with $R_2 < R_1$, then $R_F < R_2$.
  - I.e., the forward rate is less than the T_2 zero rate.

Forward Rates and Forward Rate Agreement
Forward Rates

Definition

The **instantaneous forward rate** for a maturity T is the forward rate that applies for a very short time period starting at T.

It is

,

where R is the T-year rate.

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

Definition

A **forward rate agreement** (FRA) is an OTC agreement that a certain rate will apply to a certain principal during a certain future time period.

- Consider a forward rate agreement where company X is agreeing to lend money to company Y for the period of time between T_1 and T_2.
  - R_K: The rate of interest agreed to in the FRA (agreed **today**).
  - R_F: The *forward* LIBOR interest rate calculated for the period between times T_1 and T_2 (calculated **today**)
  - R_M: The *actual* LIBOR interest rate observed for the period between times T_1 and T_2 (observed in the **future**: at time T_1).
  - L: The principal underlying the contract.
- Normally company X would earn R_M from the LIBOR loan. The FRA means that it will earn R_K.

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

- The interest rate is set at time T_1 and paid at time T_2.
- Assume that the rates R_K, R_F, and R_M are all measured with a compounding frequency reflecting the lenght of the period to which they apply.
- The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $(R_K - R_M)$.
  - It leads to a cash flow to company X at time T_2 of
  - and to company Y at time T_2 of

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

- There is another interpretation of the FRA.
  - It is an agreement where company X will receive interest on the principal between T_1 and T_2 at the fixed rate of R_K and pay interest at the realized LIBOR rate of R_M.
  - Company Y will pay interest on the principal between T_1 and T_2 at the fixed rate of R_K and receive interest at R_M.

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

- Usually FRAs are settled at time T_1 rather than T_2. The payoff must then be discounted from time T_2 to T_1.
  - For company X, the payoff at time T_1 is

$$\frac{L\left(R_K - R_M\right)\left(T_2 - T_1\right)}{1 + R_M(T_2 - T_1)}$$

  - For company Y, the payoff at time T_1 is

$$\frac{L\left(R_M - R_K\right)\left(T_2 - T_1\right)}{1 + R_M(T_2 - T_1)}$$

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

Question

Suppose that a company enters into a FRA that is designed to ensure it will receive a fixed rate of 5% on a principal of \$50 million for a 3-month period starting in 2 years.

What are the cash flows for the lender and borrower if 3-month LIBOR proves to be 5.8% for the 3-month period?

(We assume the interest rates are expressed with quarterly compounding, i.e. four times a year).

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

Solution

This FRA is an exchange where LIBOR is paid and 5% is received for the 3-month period.

The cash flow to the lender will be

$$L\left(R_K - R_M\right)\left(T_2 - T_1\right)$$

at the 2.25-year point with $R_K =$, $R_M =$, $L =$, $T_1 =$ *and* $T_2 =$.

This is equal to   .

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

Solution

The cash flow at the 2-year point writes as

The cash flow to the party on the opposite side of the transaction will be at the 2.25-year point or at the 2-year point.

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

- The value of a FRA is the present value of the difference between the interest that would be paid at rate R_F and the interest that would be paid at rate R_K.
  - It is usually the case that R_K is set equal to R_F when the FRA is first initiated.
  - The FRA is then worth zero but its value will evolve over time as R_F changes.
- Let us compare two FRAs.
  - The first promises that the LIBOR forward rate R_F will be received on a principal of L between times T_1 and T_2.
  - The second promises that R_K will be received on the same principal between the same two dates.
  - The two contracts are the same except for the interest payments received at time T_2.

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

Question

What is the excess of the present value of the second contract over the first?

We denote by R_2 the continously compounded riskless zero rate for a maturity T_2

Solution

The present value of the difference between these interest payments writes as

Forward Rates and Forward Rate Agreement
Forward Rate Agreement

- Because the value of the first FRA, where R_F is received, is zero, the value of the second FRA, where R_K is received, is

$$V_{FRA} = L \qquad (T_2 - T_1)e^{-R_2 T_2}$$

- Similarly, the value of a FRA where R_K is paid is

$$V_{FRA} = L \qquad (T_2 - T_1)e^{-R_2 T_2}$$

Chapter 5: Interest Rates
Outline

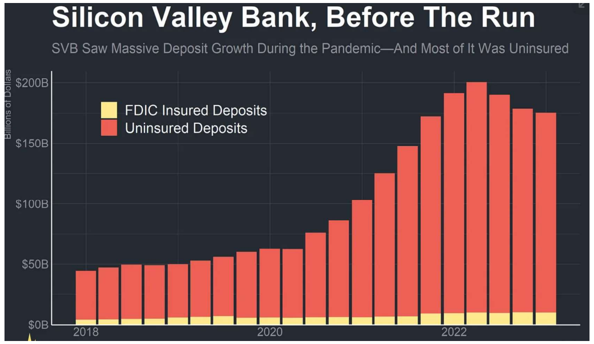
Duration and convexity
Silicon Valley Bank collapse

Role of portfolio duration and convexity in SVB collapse (March 2023)

- SVB (Silicon Valley Bank): A major bank serving tech companies.
  - 16th-largest commercial bank in the US.

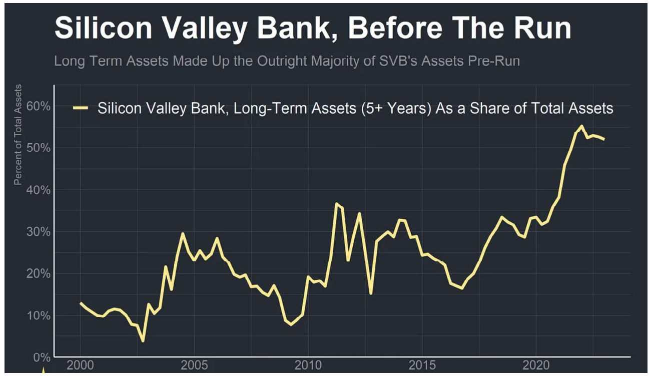
Duration and convexity
Silicon Valley Bank collapse

- Asset-Liability Mismatch: Short-term liabilities vs. long-duration assets.
  - Liability: SVB saw massive deposit growth in the early pandemic
    - ⋆ Silicon valley VC industry boomed: total deposits 2023 = 4x total deposits 2019.



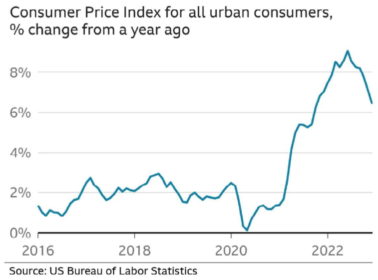
Duration and convexity
Silicon Valley Bank collapse

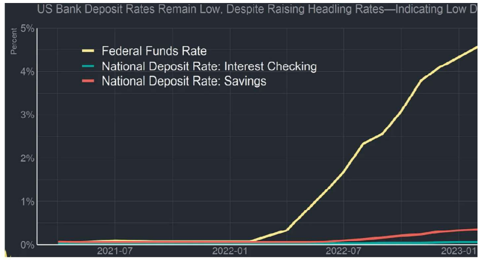
- Asset: SVB purchased unhedged long-maturity US Treasury bonds and mortgage-backed securities.
  - Long-term/total assets: from 35% (2019) to 55% (2023)



Duration and convexity
Silicon Valley Bank collapse

- Fed (2022-23) combatted inflation.: $\nearrow$ interest rates





Duration and convexity
Silicon Valley Bank collapse

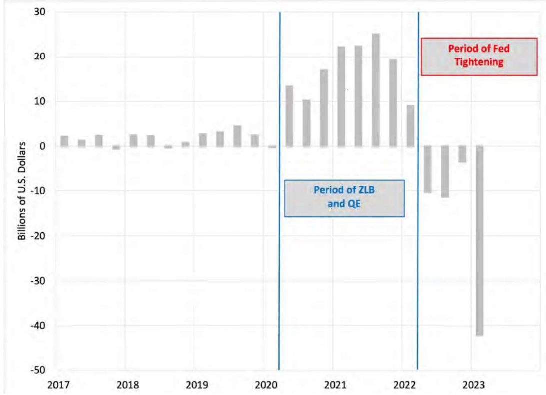
- Fed $\nearrow$ interest rates $\Longrightarrow$ $\searrow$ bond prices $\Longrightarrow$ $\searrow$SVB Asset
- SVB duration (and convexity) mismanagement
  - **Duration** indicates the years it takes to receive a bond's true cost, weighing in the present value of all future coupon and principal payments.
    - ⋆ Longer duration $\Longrightarrow$ Higher sensitivity to interest rate changes.
  - **Convexity** refers to non-linear impact of interest rate changes.
    - ⋆ Bond's price drops faster and by larger amounts than expected from linear duration effects alone.
    - ⋆ Price decline accelerated by positive convexity as rates rise.
    - ⋆ Positive convexity $\Longrightarrow$ Increased vulnerability to interest rate changes.

Duration and convexity
Silicon Valley Bank collapse

- SVB liabilities:
  - $\nearrow$interest rates $\Longrightarrow$ $\searrow$VC funding & $\nearrow$cash burn among tech companies $\Longrightarrow$ $\nearrow$ deposit outflows.

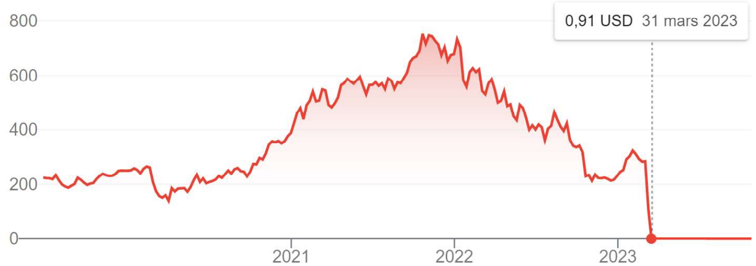
Figure 1: Silicon Valley Bank: Quarterly Change in Deposits (Billions of U.S. Dollars), 2017-1Q 2023

Source: Call Reports.



Duration and convexity
Silicon Valley Bank collapse

- SVB concentration mismanagement
  - ⋆ Assets concentrated in long-term securities
  - ⋆ Deposit base undiversified (tech companies).
- SVB Liquidity Crisis
  - Need to sell assets at a loss to meet withdrawals.
  - Liquidity issues $\Longrightarrow$ loss of depositor confidence $\Longrightarrow$fear of insolvency$\Longrightarrow$bank run.



Duration and convexity
Duration

Definition

The **duration** of a bond is a measure of how long on average the holder of the bond has to wait before receiving cash payments.

- Suppose that a bond provides the holder with cash flows c_i at time t_i, $i = 1, 2, ..., n$.
- The bond price B and bond yield y (continuously compounded) are related by

$$B = \sum_{i=1}^{n} c_i e^{-yt_i}.$$

- The duration of the bond, D, is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{B}$$

Duration and convexity
Duration

- The duration can be rewritten as

$$\sum_{i=1}^{n} t_i \frac{c_i e^{-yt_i}}{B}$$

where $\frac{c_i e^{-yt_i}}{B}$ represents the present value of the cash flow c_i to the bond price.

Duration and convexity
Duration

- The bond price is the present value of all payments.
  - The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i.
  - The sum of the weights is 1.
    - ⋆ Indeed, from $B = \sum_{i=1}^{n} c_i e^{-yt_i}$ we have $\sum_{i=1}^{n} \frac{c_i e^{-yt_i}}{B} = 1$.
- Note that for the purposes of the definition of duration all discounting is done at the bond yield rate of interest, y.
  - We do not use a different zero rate for each cash flow as we did for bond pricing.

Duration and convexity
Duration

- When a small change Δy in the yield is considered, it is approximately true that

$$\Delta B = \qquad .$$

  - (Indeed, recall that $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$, so $\Delta f(x) \simeq f'(x) \Delta x$)
- From

$$B = \sum_{i=1}^{n} c_i e^{-yt_i}$$

we have

$$\frac{dB}{dy} =$$

Duration and convexity
Duration

- so

$$\Delta B =$$

- That is

$$\frac{\Delta B}{B} =$$

Duration is important because it leads to this key relationship between the change in the yield on the bond and the change in its price.

Duration and convexity
Modified Duration

Exercise

Show that when the yield y is expressed with a compounding frequency of m times per year we have

$$\Delta B = -\frac{\Delta y B D}{1 + \frac{y}{m}}$$

Solution

Homework.

Duration and convexity
Modified Duration

- The expression

$$D^* = \frac{D}{1 + \frac{y}{m}}$$

is referred to as the the bond's **modified duration**.

  - It allows the duration relationship to be simplified to

$$\frac{\Delta B}{B} = -D^* \Delta y$$

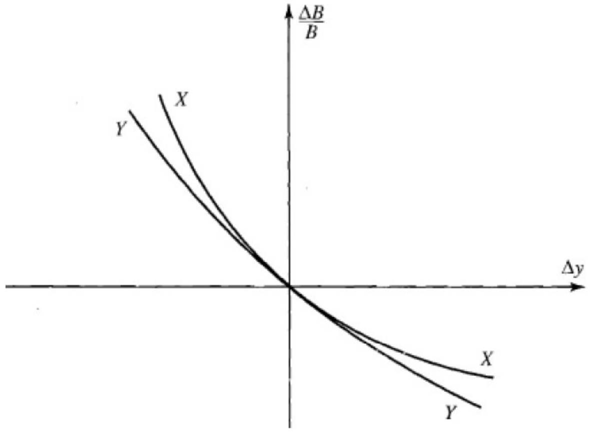
as the one we obtained with continuous compounding.

Duration and convexity
Bond Portfolios

- The duration for a bond portfolio is the weighted average duration of the bonds in the portfolio with weights proportional to prices.
- The previous key duration relationships can be used to estimate the change in the value of the bond portfolio for a small change Δy in the yields of all the bonds.
  - There is an implicit assumption that the yields of all bonds will change by approximately the same amount.
  - When the bonds have widely differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve.
- By choosing a portfolio so that the duration of assets equals the duration of liabilities (i.e., the net duration D is zero), a financial institution eliminates its exposure to small parallel shifts in the yield curve. But it is still exposed to shifts that are either large or nonparallel.

Duration and convexity
Convexity

- For large yield changes, the portfolios behave differently.
- Consider the following figure



Duration and convexity
Convexity

- Portfolio X has more curvature in its relationship with yields than portfolio Y.
  - A factor known as **convexity** measures this curvature and can be used to improve the relationship in equation.
- From Taylor series expansions, we can obtain a more accurate expression for ΔB that allows us to consider larger yield changes:

$$\Delta B =$$

Duration and convexity
Convexity

- This leads to

$$\frac{\Delta B}{B} =$$

- that is

$$\frac{\Delta B}{B} =$$

- where

$$C = \qquad =$$

denotes the curvature or **convexity** of the bond portfolio.

Duration and convexity
Convexity

- By choosing a portfolio of assets and liabilities with a net duration of zero ($D = 0$) and a net convexity of zero ($C = 0$), a financial institution can make itself immune to relatively large parallel shifts in the zero curve.
  - However, it is still exposed to nonparallel shifts.

Exercise (13)

A five-year bond with a yield of 11% (continuously compounded) pays an 8% coupon at the end of each year.

a) What is the bond's price?

b) What is the bond's duration?

c) Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.

d) Recalculate the bond's price on the basis of a 10.8% per annum yield and verify that the result is in agreement with your answer to (c).

Solution (13)

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Solution (13)

Chapter 5: Interest Rates
Outline

Theories of the Term Structure

- Why is the shape of the zero curve sometimes downward sloping, sometimes upward sloping, and sometimes partly upward sloping and partly downward sloping?
  - Three main theories have been proposed.
- **Expectations theory** conjectures that long-term interest rates should reflect expected future short-term interest rates.
  - So a forward interest rate is equal to the expected future zero interest rate.
- **Market segmentation theory** conjectures that short, medium and long rates are determined independently of each other.
  - Short (resp., medium, long)-term interest rates are determined by supply and demand in the corresponding short (resp., medium, long)-term market.
  - Markets are segmented: e.g., a large pension fund invests in bonds of a certain maturity and does not readily switch from one maturity to another.

Theories of the Term Structure

- **Liquidity preference theory** conjectures that investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers, on the other hand, usually prefer to borrow at fixed rates for long periods of time.
  - This leads to a situation in which forward rates are greater than expected future zero rates.
  - Indeed, to match the maturities of borrowers and lenders banks raise long-term rates so that forward interest rates are higher than expected future spot interest rates.
- Liquidity preference theory is the more consistent with the empirical result that yield curves tend to be upward sloping most of the time and is downward sloping only when the market expects a steep decline in short-term rates.

Chapter 5: Interest Rates
Outline

Summary

- Two important interest rates for derivative traders are Treasury rates and LIBOR rates.
  - Treasury rates are the rates paid by a government on borrowings in its own currency.
  - LIBOR rates are short-term lending rates offered by banks in the interbank market.
  - Derivatives traders have traditionally assumed that the LIBOR rate is the short-term risk-free rate at which funds can be borrowed or lent.
- The compounding frequency used for an interest rate defines the units in which it is measured.
  - Traders frequently use continuous compounding when analyzing the value of options and more complex derivatives.

Summary

- Many different types of interest rates are quoted in financial markets and calculated by analysts.
  - The n-year zero or spot rate is the rate applicable to an investment lasting for n years when all of the return is realized at the end.
  - The par yield on a bond of a certain maturity is the coupon rate that causes the bond to sell for its par value.
  - Forward rates are the rates applicable to future periods of time implied by today's zero rates.
- The method most commonly used to calculate zero rates is known as the bootstrap method.
  - It involves starting with short-term instruments and moving progressively to longer-term instruments, making sure that the zero rates calculated at each stage are consistent with the prices of the instruments.
  - It is used daily by trading desks to calculate a Treasury zero-rate curve.

Summary

- A forward rate agreement (FRA) is an OTC agreement that the LIBOR rate will be exchanged for a specified interest rate during a specified future period of time.
  - An FRA can be valued by assuming that forward LIBOR rates are realized and discounting the resulting payoff.
- An important concept in interest rate markets is duration.
  - Duration measures the sensitivity of the value of a bond portfolio to a small parallel shift in the zero-coupon yield curve.
  - Specifically

$$\Delta B = -BD\Delta y$$

where B is the value of the bond portfolio, D is the duration of the portfolio, Δy is the size of a small parallel shift in the zero curve, and ΔB is the resultant effect on the value of the bond portfolio.

Summary

- Liquidity preference theory can be used to explain the interest rate term structures that are observed in practice.
  - The theory argues that most entities like to borrow long and lend short.
  - To match the maturities of borrowers and lenders, it is necessary for financial institutions to raise long-term rates so that forward interest rates are higher than expected future spot interest rates.