Introduction to Derivative Instruments

Paris Dauphine University - Master I.E.F. (272)
Autumn 2024

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Slides on book: John C. Hull, “Options, Futures, and Other Derivatives”, Pearson ed.

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Chapter 3

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Chapter 3:

Determination of Forward and Futures Prices

Introduction

Motivation

- How forward prices and futures prices are related to the spot price of the underlying asset?
- Forward contracts are easier to analyze than futures contracts because there is no daily settlement, but only a single payment at maturity.
- We therefore start by considering the relationship between the forward price and the spot price.

Introduction

Consumption vs Investment Assets

- An **investment asset** is an asset that is held by significant numbers of people purely for investment purposes.
  - E.g., stocks, bonds, gold, silver, ...
- A **consumption asset** is an asset that is held primarily for consumption.
  - E.g., copper, oil, pork bellies, ...
- As we shall see we can use arbitrage arguments to determine the forward and futures prices of an investment asset from its spot price and other observable market variables.
- We cannot do this for consumption assets.

Introduction
Short selling

- **Short selling** involves selling securities you do not own.
- Occasionally there are temporary bans on short selling.
  - This happened in a number of countries in 2008 because it was considered that short selling contributed to the high market volatility that was being experienced.
  - The SEC introduced in February 2010 restrictions on short selling.
    - When the price of a stock has decreased by more than 10% in one day, there are restrictions on short selling for that day and the next.
- Your broker borrows the securities from another client and sells them in the market in the usual way.
- At some stage you must buy the securities so they can be replaced in the account of the client.

Introduction
Short selling

Example

In April, you borrow 500 shares and sell them for \$120 ($= \$60,000$)

In July, you buy 500 shares for \$100 per share, and replace borrowed shares to close short position ($= -\$50,000$)

Net profit is $\$60,000 - \$50,000 = \$10,000$.

Introduction
Notation for Valuing Futures and Forward Contracts

- S_0: Spot price today;
- F_0: Futures or forward price today;
- T: Time (in years) until delivery date;
- r: Risk-free interest rate for maturity T.
  - i.e., the rate at which money is borrowed or lent when there is no credit risk.

Forward Price for an Investment Asset
An Arbitrage Opportunity?

Question

Suppose that:

- The spot price of a non-dividend-paying stock is \$40
- The 3-month forward price is \$43
- The 3-month US\$ interest rate is 5% per annum

Is there an arbitrage opportunity?

Forward Price for an Investment Asset
An Arbitrage Opportunity?

Solution

Forward Price for an Investment Asset
Another Arbitrage Opportunity?

Question

Suppose now that the 3-month forward price is US$39.

Is there an arbitrage opportunity?

Forward Price for an Investment Asset
Another Arbitrage Opportunity?

Solution

Forward Price for an Investment Asset
Generalization

- Using our notation, the no-arbitrage condition on the forward price writes as
$$F_0 = S_0 e^{rT}$$
- Indeed, if $F_0 > S_0 e^{rT}$, arbitrageurs can buy the asset and short forward contracts on the asset.
- If $F_0 < S_0 e^{rT}$ they can short the asset and enter into long forward contracts on it.

When an Investment Asset Provides a Known Income

- Consider a long forward contract to purchase a coupon-bearing bond whose current price is \$900.
- We will suppose that the forward contract matures in 9 months.
- We will also suppose that a coupon payment of \$40 is expected after 4 months.
- We assume that the 4-month and 9-month risk-free interest rates (continuously compounded) are, respectively, 3% and 4% per annum.
- What is the forward price today, F_0?

When an Investment Asset Provides a Known Income

- To answer this question let us first simplify the problem.
- What would be the forward price today if there was no coupon payment?
  - Applying our previous formulae we obtain

$$F_0 = S_0 e^{rT} = 900e^{0.04\times\frac{9}{12}} \simeq 927.41$$

- What would be the forward price today if the coupon payment was expected immediately after the purchase?
  - $F_0 = (S_0 - 40)\, e^{rT} = 860e^{0.04\times\frac{9}{12}} \simeq 886.19$

When an Investment Asset Provides a Known Income

- What would be the forward price today if the coupon payment was expected after 9 months?
  - The present value of the coupon is $40e^{-rT} = 40e^{-0.04\times\frac{9}{12}} \simeq 38.82$
  - So, $F_0 = (S_0 - 38.82)\, e^{rT} = 861.18e^{0.04\times\frac{9}{12}} \simeq 887.41$
- Now, what is the answer to the initial question?
  - The present value of the coupon is $40e^{-0.03\times\frac{4}{12}} \simeq 39.6$
  - So, $F_0 = (S_0 - 39.6)\, e^{rT} = 860.4e^{0.04\times\frac{9}{12}} \simeq 886.6$

When an Investment Asset Provides a Known Income
Generalization

- When an investment asset will provide income with a present value of I during the life of a forward contract, we have

$$F_0 = (S_0 - I)\, e^{rT}.$$

- If $F_0 > (S_0 - I)\, e^{rT}$, an arbitrageur can lock in a profit by buying the asset and shorting a forward contract on the asset.
- if $F_0 < (S_0 - I)\, e^{rT}$, an arbitrageur can lock in a profit by shorting the asset and taking a long position in a forward contract.
- If short sales are not possible, investors who own the asset will find it profitable to sell the asset and enter into long forward contracts.

When an Investment Asset Provides a Known Income
An Arbitrage Opportunity?

Question

Suppose the previous 9-months forward price is out of line with spot price.

What is the arbitrage opportunity if the forward price is 910?

When an Investment Asset Provides a Known Income
An Arbitrage Opportunity?

Solution

When an Investment Asset Provides a Known Income
Another Arbitrage Opportunity?

Question

What is the arbitrage opportunity if the forward price is 870?

When an Investment Asset Provides a Known Income
Another Arbitrage Opportunity?

Solution

When an Investment Asset Provides a Known Yield

- Consider a 6-month forward contract on an asset that is expected to provide income equal to 2% of the asset price once during a 6-month period.
- The risk-free rate of interest (with continuous compounding) is 10% per annum. The asset price is \$25.
- The yield is 4% per annum (with continuous compounding).
- What is the forward price, F_0?
- The forward price is $F_0 = 25e^{(0.1-0.04)\frac{1}{2}} \simeq 25.76$

When an Investment Asset Provides a Known Yield
Generalization

- When q is the average yield per annum on an asset during the life of a forward contract (expressed with continuous compounding), we have
$$F_0 = S_0 e^{(r-q)T}.$$
- Note that the NPV of the dividend writes as
$$S_0(1 - e^{-qT})$$
  - Indeed, when the dividend is given at date 0, it is worth: $S_0 - S_\varepsilon = S_0 - S_0 e^{-q\varepsilon}$ (with $\varepsilon > 0$).
  - More generally, $S_0 e^{-qT}$ denote the value of the asset (S_0) minus the NPV of the dividend.

Valuing a Forward Contract

- We denote:
  - F_0: the forward price for a contract that would be negotiated today;
  - K: the delivery price; and
  - f: the value of the forward contract today.
- At the beginning of the life of the forward contract, $F_0 = K$ and $f = 0$.
- As time passes, K stays the same (fixed by the contract), but the forward price F_0 changes and the value of the contract f becomes either positive or negative.
- We can value a long forward contract on an asset by making the assumption that the forward price F_0 equals the price of the asset at the maturity of the forward contract.

Valuing a Forward Contract

- A long forward contract on a non-dividend-paying stock was entered into some time ago.
  - It currently has 6 months to maturity.
  - The risk-free rate of interest (with continuous compounding) is 10% per annum, the stock price is \$25, and the delivery price is \$24.
- What is the value of the forward contract, denoted as f?
  - The 6-month forward price is $F_0 = 25e^{0.1\frac{1}{2}} \simeq 26.28$
  - The value of the forward contract is $f = 25 - 24e^{-0.1\frac{1}{2}} = (26.28 - 24)e^{-0.1\frac{1}{2}} \simeq \$2.17$

Valuing a Forward Contract
Generalization

- When K is the delivery price and F_0 is the forward price for a contract that would be negotiated today, the value of a long forward contract is

$$f = (F_0 - K)e^{-rT}$$

Intuition

Imagine there is a long forward contract that has a delivery price of F_0.

The difference between this contract and our contract with delivery price K is only in the amount that will be paid for the underlying asset at time T.

The cash outflow difference of $F_0 - K$ at time T translates to a difference of $(F_0 - K)e^{-rT}$ today.

Valuing a Forward Contract
Generalization

- Similarly, the value of a short forward contract with delivery price K is

$$f = (K - F_0)e^{-rT}$$

Exercise (3)

A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free rate of interest is 10% per annum with continuous compounding.

a) What are the forward price and the initial value of the forward contract?

b) Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

Solution (3)

Exercise (3)

A one-year long forward contract on a non-dividend-paying stock is entered into when the stock price is \$40 and the risk-free rate of interest is 10% per annum with continuous compounding.

b) Six months later, the price of the stock is \$45 and the risk-free interest rate is still 10%. What are the forward price and the value of the forward contract?

Solution (3)

Forward vs Futures Prices

- When the maturity and asset price are the same, forward and futures prices are usually assumed to be equal.
- In practice, there are a number of factors not reflected in theoretical models that may cause forward and futures prices to be different.
  - These include taxes, transactions costs, and the treatment of margins.
  - Futures contracts are more liquid and easier to trade than forward contracts.
  - The risk that the counterparty will default may be less in the case of a futures contract because of the role of the exchange clearinghouse.
  - Variation of the interest rate has an impact on what can be earned from margins of a future contracts (daily settlement).

Forward vs Futures Prices

- This last effect (interest rate) is accentuated when the price S of the underlying asset is strongly positively correlated with interest rates.
  - A strong positive correlation between interest rates and the asset price implies that when S increases, an investor who holds a long futures position makes an immediate gain because of the daily settlement procedure.
    - The positive correlation indicates that it is likely that interest rates have also increased.
    - The gain will therefore tend to be invested at a higher than average rate of interest.
    - Similarly, when S decreases, the investor will incur an immediate loss.
    - This loss will tend to be financed at a lower than average rate of interest.
  - An investor holding a forward contract rather than a futures contract is not affected in this way by interest rate movements.

Futures Prices of Stock Indices

- A stock index can usually be regarded as the price of an investment asset that pays dividends.
  - The investment asset is the portfolio of stocks underlying the index;
  - and the dividends paid by the investment asset are the dividends that would be received by the holder of this portfolio.
- It is usually assumed that the dividends provide a known yield rather than a known cash income.
- If q is the dividend yield rate, the futures price, F_0, writes as

$$F_0 = S_0 e^{(r-q)T}$$

- For the formula to be true it is important that the index represent an investment asset.
- In other words, changes in the index must correspond to changes in the value of a tradable portfolio.

Futures Prices of Stock Indices
Index Arbitrage

- When $F_0 > S_0 e^{(r-q)T}$ an arbitrageur buys the stocks underlying the index and sells futures.
- When $F_0 < S_0 e^{(r-q)T}$ an arbitrageur buys futures and shorts or sells the stocks underlying the index.

Futures and Forwards on Currencies

- A foreign currency is analogous to a security providing a yield.
- The yield is the foreign risk-free interest rate, denoted as r_f.
- When S_0 is the current spot price in US dollars of one unit of the foreign currency and F_0 is the forward or futures price in US dollars of one unit of the foreign currency, we have

$$F_0 = S_0 e^{(r - r_f)T}$$

Futures and Forwards on Currencies

Intuition

There are two ways of converting 1 unit of a foreign currency to dollars at time T.

Here, S_0 is spot exchange rate, F_0 is forward exchange rate, and r and r_f are the dollar and foreign risk-free rates.

First way. 1 unit of a foreign currency at time zero gives $e^{r_f T}$ units of foreign currency at time T which in turns, give $e^{r_f T} F_0$ dollars at time T.

Second way. 1 unit of a foreign currency at time zero gives S_0 dollars at time zero which in turns, give $e^{rT} S_0$ dollars at time T.

Exercise (7)

The two-month interest rates in Switzerland and the United States are 2% and 5% per annum, respectively, with continuous compounding. The spot price of the Swiss franc is \$0.8000. The futures price for a contract deliverable in two months is \$0.8100.

What arbitrage opportunities does this create?

Solution (7)

Solution (7)

Action now:

-

-

-

Action in 2 months:

-

-

-

Arbitrage profit:

Consumption Assets

Storage is Negative Income

- When U is the present value of all the storage costs, net of income, during the life of a forward contract, we have

$$F_0 = (S_0 + U)\, e^{rT}$$

- If the storage costs (net of income) incurred at any time are proportional to the price of the commodity, they can be treated as negative yield.
- When u denotes the storage costs per annum as a proportion of the spot price net of any yield earned on the asset, we have

$$F_0 = S_0 e^{(r+u)T}$$

Consumption Assets

Convenience Yields

- Users of a consumption commodity may feel that ownership of the physical commodity provides benefits that are not obtained by holders of futures contracts.
  - For example, an oil refiner is unlikely to regard a futures contract on crude oil in the same way as crude oil held in inventory.
    - The crude oil in inventory can be an input to the refining process, whereas a futures contract cannot be used for this purpose.
- The benefits from holding the physical asset are sometimes referred to as the **convenience yield** provided by the commodity.
- The greater the possibility that shortages will occur, the higher the convenience yield.
  - If inventories are low, shortages are more likely and the convenience yield is usually higher.
- When y denotes the convenience yield, we have

$$F_0 = S_0 e^{(r+u-y)T}$$

Futures Prices and Spot Prices

The Cost of Carry

- The relationship between futures prices and spot prices can be summarized in terms of the **cost of carry**.
  - This measures the storage cost plus the interest that is paid to finance the asset less the income earned on the asset.
  - For a non-dividend-paying stock, the cost of carry is r, because there are no storage costs and no income is earned;
  - for a stock index, it is $r - q$, because income is earned at rate q on the asset.
  - for a currency, it is $r - r_f$;
  - for a commodity that provides income at rate q and requires storage costs at rate u, it is $r - q + u$;
  - and so on.

Futures Prices and Spot Prices

The Cost of Carry

- Define the cost of carry as c.
- For an investment asset, the futures price is

$$F_0 = S_0 e^{cT}$$

- For a consumption asset, it is

$$F_0 = S_0 e^{(c-y)T} \leq S_0 e^{cT}.$$

Futures Prices and Spot Prices
Futures Prices & Expected Future Spot Prices

- The **expected spot price** of an asset at a certain future time is the market's average opinion about what the spot price of that asset will be at that time.
  - As we have seen in Chapter 2, the futures price converges to the spot price at maturity.

Futures Prices and Spot Prices
Futures Prices & Expected Future Spot Prices

- Relationship between futures price F_0 and expected future spot price $\mathbb{E}(S_T)$, where $\mathbb{E}(\cdot)$ denotes expected value:
  - When there is **no systematic risk**, the futures price is an unbiased estimate of the expected future spot price

$$F_0 = \mathbb{E}(S_T)$$

    - Nonsystematic risk should not be important to an investor because it can be almost completely eliminated by holding a well-diversified portfolio.

Futures Prices and Spot Prices
Futures Prices & Expected Future Spot Prices

- Relationship between futures price F_0 and expected future spot price $\mathbb{E}(S_T)$, where $\mathbb{E}(\cdot)$ denotes expected value:
  - When the asset underlying the futures contract has **positive systematic risk** (e.g., a stock index), we should expect the futures price to understate the expected future spot price

$$F_0 < \mathbb{E}(S_T)$$

Futures Prices and Spot Prices
Futures Prices & Expected Future Spot Prices

- Relationship between futures price F_0 and expected future spot price $\mathbb{E}(S_T)$, where $\mathbb{E}(\cdot)$ denotes expected value:
  - When the asset underlying the futures contract has **negative systematic risk** (e.g., gold), we should expect the futures price to overstate the expected future spot price

$$F_0 > \mathbb{E}(S_T)$$

Futures Prices and Spot Prices
Futures Prices & Expected Future Spot Prices

- Suppose k is the expected return required by investors in an asset. We then have
$$S_0 = \mathbb{E}(S_T)e^{-kT}$$
- The no-arbitrage condition writes as
$$F_0 = S_0e^{rT} = \mathbb{E}(S_T)e^{(r-k)T}$$
  - When there is no systematic risk, we have $k = r$.
  - When the asset underlying the futures contract has positive systematic risk, we have $k > r$.
  - When the asset underlying the futures contract has negative systematic risk, we have $k < r$.

Summary

- For most purposes, the futures price of a contract with a certain delivery date can be considered to be the same as the forward price for a contract with the same delivery date.
  - It can be shown that in theory the two should be exactly the same when interest rates are perfectly predictable.
- It is convenient to divide futures contracts into two categories:
  - those in which the underlying asset is held for investment by a significant number of investors; and
  - those in which the underlying asset is held primarily for consumption purposes.

Summary

- In the case of investment assets, we have considered three different situations:
  - 1. The asset provides no income.
    - ⋆ Forward/futures is S_0e^{rT}
    - ⋆ Value of long forward contract price with delivery price K is $S_0 - Ke^{-rT}$
  - 2. The asset provides a known dollar income.
    - ⋆ Forward/futures is $(S_0 - I)\,e^{rT}$
    - ⋆ Value of long forward contract price with delivery price K is $S_0 - I - Ke^{-rT}$
  - 3. The asset provides a known yield.
    - ⋆ Forward/futures is $S_0e^{(r-q)T}$
    - ⋆ Value of long forward contract price with delivery price K is $S_0e^{-qT} - Ke^{-rT}$

Summary

- In the case of consumption assets, it is not possible to obtain the futures price as a function of the spot price and other observable variables.
  - Here the parameter known as the asset's convenience yield becomes important.
    - ⋆ It measures the extent to which users of the commodity feel that ownership of the physical asset provides benefits that are not obtained by the holders of the futures contract.
    - ⋆ These benefits may include the ability to profit from temporary local shortages or the ability to keep a production process running.
- The cost of carry is the storage cost of the underlying asset plus the cost of financing it minus the income received from it.
  - In the case of investment assets, the futures price is greater than the spot price by an amount reflecting the cost of carry.
  - In the case of consumption assets, the futures price is greater than the spot price by an amount reflecting the cost of carry net of the convenience yield.

Summary

- The relationship between the futures price and the expected future spot price depends on whether the return on the asset is positively or negatively correlated with the return on the stock market.
  - Positive (resp. negative) correlation will tend to lead to a futures price lower (resp. higher) than the expected future spot price.
  - Only when the correlation is zero will the theoretical futures price be equal to the expected future spot price.

Underlying asset	*Relationship of expected return k from asset to risk-free rate r*	*Relationship of futures price F to expected future spot price $E(S_T)$*
No systematic risk	$k = r$	$F_0 = E(S_T)$
Positive systematic risk	$k > r$	$F_0 < E(S_T)$
Negative systematic risk	$k < r$	$F_0 > E(S_T)$