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# Bertrand equilibrium with subadditive costs

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ABSTRACT

We show here, in contrast to recent results, that if firms have different cost functions (that are strictly subadditive), such that the 'monopoly breakeven prices' are different, then in a homogeneous product duopoly there is always a Bertrand equilibrium (either in pure strategies or in mixed strategies).



## 1. Introduction

We consider a simultaneous move price choice game in a homogeneous product oligopoly. The firm quoting the lowest price gets the entire demand. If there is a tie at the lowest price, then we assume that all the firms tied at the lowest price share the demand equally. We will also assume that a firm always meets the demand that it faces at the posted price.[1]

In this framework, when firms have *identical cost functions*, the following results have recently been proved. With linear demands, Baye and Kovenock (2008) and Hoernig (2007, the example in Section 4), demonstrate that if there are positive fixed costs and if the marginal costs are constant, then Bertrand equilibrium does not exist either in pure or in mixed strategies. Saporiti and Coloma (2010) have shown that if there is a pure strategy Bertrand equilibrium, then the total cost function is not strictly subadditive at every output greater than or equal to the demand at the oligopoly breakeven price. From this one can logically deduce that if the cost function is strictly subadditive on $[0, \infty)$ then there cannot be a pure strategy Bertrand equilibrium. Dastidar (in press) generalises and extends these results and shows that if costs are strictly subadditive on $[0, \infty)$ then there exists *no equilibrium either in pure strategies or in mixed strategies*.[2]

In this short paper we prove the following result. If costs are strictly subadditive and firms have *different* cost functions, such that the monopoly breakeven prices are different, then *there is always an equilibrium* (either in pure strategies or in mixed strategies). When there is a mixed strategy equilibrium, the structure of the equilibrium is same as in Blume (2003). We will show our result in the context of a duopoly. But this can be easily extended to the case of an oligopoly. The point is that even with strict subadditivity, a little bit of asymmetry in costs restores the existence of a Bertrand equilibrium.

## 2. The model

Consider a simultaneous move price choice game in a homogeneous product, *asymmetric* cost duopoly. The demand function is given by $D : [0, \infty) \longrightarrow [0, \infty)$ and the cost function for firm $i$ is

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[1] Here we closely follow page 118 of Vives (1999) to justify our assumption. This assumption can be rationalised by assuming that when a firm sets a price $P_i$, this represents a commitment to supply the forthcoming demand. This may be the case in regulated industries (for example, in the supply of electricity or telephone) or the result of consumer protection laws. For example, it is typical of "common carrier" regulation, requiring firms to meet all demand at the set prices. If the supply of a product is exhausted the customer may take a "rain check" (a coupon with which to purchase the good at the posted price at a later date).

[2] We report here some results with non-subadditive costs. Dastidar (in press) has shown that if all firms have identical costs and costs are strictly superadditive then there is always a pure strategy Bertrand equilibrium. With convex variable costs, Saporiti and Coloma (2010) have shown that if the total cost function is not strictly subadditive at the output corresponding to the 'oligopoly breakeven price' then there is a pure strategy Bertrand equilibrium.

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given by $C_i : [0, \infty) \longrightarrow [0, \infty)$. Each firm's cost function takes the following form:

$$C_i(x) = \begin{cases} 0 & \text{if } x = 0 \\ F_i + V_i(x) & \text{if } x > 0, \end{cases}$$

where $F_i \geq 0$ is the fixed cost of production and $V_i : [0, \infty) \longrightarrow [0, \infty)$ is the variable cost function.

We now introduce the definition of subadditivity.

**Definition 1.** A cost function $C : [0, \infty) \longrightarrow [0, \infty)$ is strictly subadditive on $[0, \infty)$ if and only if $C(x + y) < C(x) + C(y)$ for all $x, y \in (0, \infty)$.

We make the following assumptions.

**Assumption 1.** $\exists$ finite positive numbers $P^{\max}$ and $Q^{\max}$ such that $D(P) = 0\ \forall P \geq P^{\max}$ and $D(0) = Q^{\max}$. $D(P)$ is continuous over $[0, \infty)$ and $D(P)$ is twice continuously differentiable over $(0, P^{\max})$. Also $D'(P) < 0, \forall P \in (0, P^{\max})$.

**Assumption 2.** For all $i$, $C_i(x)$ is strictly subadditive on $[0, \infty)$. Also, $C_i(x)$ is twice continuously differentiable over $(0, \infty)$. $V_i(0) = 0$ and $C_i'(x) > 0\ \forall x \in (0, \infty)$.

**Assumption 3.** The firm which quotes the lowest price gets all the demand. Any firm which quotes a price higher than its rival gets no demand. If there is a tie at any price, the two firms share the demand equally.

**Assumption 4.** We assume that in price competition firms have to meet the demand that they face at the posted price.

Define for all $i$

$$\pi_i(P) = PD(P) - C_i(D(P))$$

$$\text{and } \hat{\pi}_i(P) = \frac{1}{2}PD(P) - C_i\left(\frac{1}{2}D(P)\right).$$

Clearly $\pi_i(P)$ is the profit going to firm $i$ when it serves the market alone and $\hat{\pi}_i(P)$ is the profit when it shares the market equally with the other firm. Given our assumptions on demand and cost, both $\pi_i(.)$ and $\hat{\pi}_i(.)$ are continuous over $[0, P^{\max})$ and twice continuously differentiable over $(0, P^{\max})$. Also $\pi_i(P^{\max}) = \hat{\pi}_i(P^{\max}) = 0$.

Suppose that $P_i^{\text{mon}} = \arg \max_{P \in [0, P^{\max}]} \pi_i(P)$.

It may be noted that $P_i^{\text{mon}}$ is the profit maximising monopoly price of a firm. We now provide our last assumption which ensures that $P_i^{\text{mon}}$ is unique.

**Assumption 5.** For all $i$, $\pi_i(.)$ and $\pi_i'(.)$ are bounded and $\frac{\partial^2 \pi_i(.)}{\partial P^2} < 0$ for all $P \in (0, P^{\max})$. Also $\pi_i(P_i^{\text{mon}}) > 0$.

It may be noted that all our assumptions are standard.

### 2.1. Some more notation and definitions

Note that $\pi_i(.)$ is continuous in $P$ on $[0, P^{\max})$. Since $\pi_i(0) = -C_i(D(0)) = -C_i(Q^{\max}) < 0$ and $\pi_i(P_i^{\text{mon}}) > 0$ (from Assumption 5), $\exists P$ s.t. $\pi_i(P) = 0$.

Suppose that $\tilde{P}_i = \min\{P \mid \pi_i(P) = 0,\ P \in [0, P^{\max}]\}$.

We call $\tilde{P}_i$ the '*monopoly breakeven price*' for firm $i$.

Our assumptions ensure that $\tilde{P}_i < P_i^{\text{mon}}$ for all $i$.

## 3. The results

Before proceeding to our main set of results we first provide the following preliminary results.

**Lemma 1.** *For all $i$ and for all $P \in (0, P^{\max})$, $\pi_i(P) > 2\hat{\pi}_i(P)$.*

**Proof.**

$$\pi_i(P) - \hat{\pi}_i(P) = PD(P) - C_i(D(P)) - \frac{1}{2}PD(P) + C_i\left(\frac{1}{2}D(P)\right)$$

$$= \frac{1}{2}PD(P) - \left[C_i(D(P)) - C_i\left(\frac{1}{2}D(P)\right)\right]. \quad (1)$$

Strict subadditivity of $C_i(x)$ implies that

$$C_i(D(P)) < 2C_i\left(\frac{1}{2}D(P)\right)$$

$$\Longrightarrow C_i(D(P)) - C_i\left(\frac{1}{2}D(P)\right) < C_i\left(\frac{1}{2}D(P)\right). \quad (2)$$

Using (2) in (1) we get

$$\pi_i(P) - \hat{\pi}_i(P) > \frac{1}{2}PD(P) - C_i\left(\frac{1}{2}D(P)\right) = \hat{\pi}_i(P)$$

$$\Longrightarrow \pi_i(P) > 2\hat{\pi}_i(P). \quad \square$$

**Lemma 2.** *For all $i$, $\pi_i'(P) > 0$ for all $P \in (0, P_i^{\text{mon}})$ and $\pi_i'(P) < 0$ for all $P \in (P_i^{\text{mon}}, P^{\max})$.*

**Proof.** Straightforward. Follows from Assumption 5. $\square$

First note that since for all $i$ we have $\tilde{P}_i < P_i^{\text{mon}}$ we cannot have a situation where both $\tilde{P}_1 \geq P_2^{\text{mon}}$ and $\tilde{P}_2 \geq P_1^{\text{mon}}$ hold true. Our first result deals with the case where one of them holds true.

**Proposition 1.** *If either $\tilde{P}_1 \geq P_2^{\text{mon}}$ or if $\tilde{P}_2 \geq P_1^{\text{mon}}$, then there is a pure strategy Bertrand equilibrium.*

**Proof.** To demonstrate this result let us take the case where $\tilde{P}_2 \geq P_1^{\text{mon}}$. For this case the following strategy profile constitutes a pure strategy Bertrand equilibrium:

$$P_1^* = P_1^{\text{mon}} \quad \text{and} \quad P_2^* \in (P_1^{\text{mon}}, \infty).$$

It is obvious that firm 1 does not gain by deviating. Firm 2 gets zero in equilibrium. If it quotes $P_1^{\text{mon}}$ it gets $\hat{\pi}_2(P_1^{\text{mon}})$. From Lemma 1 we know that $\hat{\pi}_2(P_1^{\text{mon}}) < \frac{1}{2}\pi_2(P_1^{\text{mon}})$ and since $\tilde{P}_2 \geq P_1^{\text{mon}}$ we have $\pi_2(P_1^{\text{mon}}) \leq 0$ and this implies $\hat{\pi}_2(P_1^{\text{mon}}) < 0$. Therefore, 2 does not gain by quoting $P_1^{\text{mon}}$. If 2 quotes any price $P$ which is strictly lower than $P_1^{\text{mon}}$ it gets $\pi_2(P) < \pi_2(P_1^{\text{mon}})$. This is because $P_1^{\text{mon}} \leq \tilde{P}_2 < P_2^{\text{mon}}$ and because of the fact that $\pi_2(P)$ is strictly increasing in $P$ for all $P \in (0, P_2^{\text{mon}})$ (see Lemma 2). Hence 2 does not gain by quoting a price strictly lower than $P_1^{\text{mon}}$ and consequently, the stated strategy profile constitutes a pure strategy Bertrand equilibrium. $\square$

As noted before, we cannot have a situation where both $\tilde{P}_1 \geq P_2^{\text{mon}}$ and $\tilde{P}_2 \geq P_1^{\text{mon}}$ hold true. Proposition 1 dealt with the case where one of them holds true. In our last main result we deal with the case where neither holds true.

**Proposition 2.** *If $\tilde{P}_1 \neq \tilde{P}_2$, $\tilde{P}_1 < P_2^{\text{mon}}$ and $\tilde{P}_2 < P_1^{\text{mon}}$, then there is no pure strategy Bertrand equilibrium. However, there is always a mixed strategy Bertrand equilibrium.*

**Proof.** We first prove the non-existence of any pure strategy equilibrium.

If possible let $\left(P_1^*, P_2^*\right)$ be a pure strategy Bertrand equilibrium. There are now two possible cases.

Case (1) $P_1^* = P_2^* = P^*$ (say). Case (2) $P_1^* \neq P_2^*$.

First take Case 1. Here in equilibrium firm $i$ gets $\hat{\pi}_i(P^*)$. Also, $\hat{\pi}_i(P^*) \geq 0$. Otherwise, a firm can deviate by quoting a higher price than its rival and get zero. We now claim that $P^* > \tilde{P}_i$ (where $i = 1, 2$). To show this, note the following. Suppose on the contrary $P^* \leq \tilde{P}_i$. Then we have $\pi_i(P^*) \leq 0$ (from Lemma 2 and the definition of $\tilde{P}_i$). From Lemma 1 we have $\hat{\pi}_i(P^*) < \frac{1}{2}\pi_i(P^*) \leq 0$ but this contradicts the fact that in equilibrium $\hat{\pi}_i(P^*) \geq 0$. Hence $P^* > \tilde{P}_i$ (where $i = 1, 2$). From Lemma 1 this means that $\pi_i(P^*) > 2\hat{\pi}_i(P^*) \geq 0$. But then any firm $i$ can deviate by undercutting its rival and choosing $P^* - \varepsilon$ (where $\varepsilon$ is arbitrarily small) and getting $\pi_i(P^* - \varepsilon) > \hat{\pi}_i(P^*)$. Hence, Case 1 cannot arise in equilibrium.

Now take Case 2. Without loss of any generality suppose that $P_1^* < P_2^*$. In equilibrium, firm 1 gets $\pi_1(P_1^*)$ and firm 2 gets zero. There are now two possible subcases, subcase (i): $P_1^* > \tilde{P}_2$ and subcase (ii): $P_1^* \leq \tilde{P}_2$.

For subcase (i) firm 2 can deviate and quote a price $P_1^* - \varepsilon$ s.t. $P_1^* > P_1^* - \varepsilon > \tilde{P}_2$ and get $\pi_2(P_1^* - \varepsilon) > 0$. Hence, this subcase cannot arise in equilibrium.

For subcase (ii) we have $P_1^* \leq \tilde{P}_2 < P_1^{\text{mon}}$ (see the hypotheses of the proposition). Note that in equilibrium firm 1 gets $\pi_1(P_1^*)$. It can deviate by quoting a price $P_1^* + \varepsilon$ s.t. $P_2^* > P_1^* + \varepsilon > P_1^*$ and get $\pi_1(P_1^* + \varepsilon) > \pi_1(P_1^*)$. This follows as $\pi_i(P)$ is strictly increasing for all $P \in (0, P_i^{\text{mon}})$ (Lemma 2). Hence this subcase cannot arise in equilibrium.

We now closely follow Blume (2003) to demonstrate the existence of mixed strategy Bertrand equilibrium. Without loss of generality suppose that $\tilde{P}_1 < \tilde{P}_2$. The mixed strategy equilibrium is as follows. Firm 1 quotes $P_1^* = \tilde{P}_2$ and firm 2 randomises uniformly over $[\tilde{P}_2, \tilde{P}_2 + a]$ where $a$ is *small enough*. Here the distribution function of 2's strategy is given by $F(x) = \frac{x - \tilde{P}_2}{a}$ and the corresponding density function is $f(x) = \frac{1}{a}$. We now show that it is indeed an equilibrium.

Given firm 1's strategy, this is firm 2's best response. It gets zero in equilibrium and it cannot get a higher payoff by deviating.

In equilibrium, firm 1 gets $\pi_1(\tilde{P}_2) > 0$ (since $\tilde{P}_1 < \tilde{P}_2 < P_1^{\text{mon}}$). Again, since $\tilde{P}_2 < P_1^{\text{mon}}$ firm 1 does not gain by quoting a lower price (since from Lemma 2 we have that $\pi_1(P)$ is strictly increasing in $P$ for all $P \in (0, P_1^{\text{mon}})$). If firm 1 quotes a price strictly higher than $\tilde{P}_2 + a$ it will get zero (as 2 undercuts it with probability 1). We now show that firm 1 does not gain by posting a price in $(\tilde{P}_2, \tilde{P}_2 + a]$. If firm 1 posts a price $P$ in $(\tilde{P}_2, \tilde{P}_2 + a]$ it undercuts its rival with probability $(1 - F(P))$ and its expected payoff is

$$E_1 = \pi_1(P)(1 - F(P)) = \pi_1(P)\left(1 - \frac{P - \tilde{P}_2}{a}\right).$$

Now

$$\frac{dE_1}{dP} = -\frac{\pi_1(P)}{a} + \left(1 - \frac{P - \tilde{P}_2}{a}\right)\pi_1'(P).$$

Note that for all $P \in (\tilde{P}_2, \tilde{P}_2 + a]$, $\left(1 - \frac{P - \tilde{P}_2}{a}\right) \in [0, 1)$. Also, $\pi_1'(P)$ is bounded (Assumption 5) and $\pi_1(P) > 0$ for all $P \in (\tilde{P}_1, P_1^{\text{mon}})$ and also note that $\tilde{P}_2 \in (\tilde{P}_1, P_1^{\text{mon}})$. Hence, for *small enough a* we have $\frac{dE_1}{dP} < 0$. This demonstrates that firm 1 does not prefer posting a price in $(\tilde{P}_2, \tilde{P}_2 + a]$. Therefore, the proposed mixed strategy profile constitutes a Bertrand equilibrium. □

## 4. Conclusion

It has been recently shown that in a homogeneous product Bertrand oligopoly with identical and strictly subadditive costs there exists no equilibrium, either in pure strategies or in mixed strategies. We have demonstrated that if firms have different cost functions and if the 'monopoly breakeven prices' are different then there is always a Bertrand equilibrium (either in pure strategies or in mixed strategies). As noted in the introduction, the main point is that even with strict subadditivity, a little bit of asymmetry in costs restores the existence of a Bertrand equilibrium.

## References


Baye, M.R., Kovenock, D., 2008. Bertrand competition. In: Durlauf, S.N., Blume, L.E. (Eds.), The New Palgrave Dictionary of Economics, second ed. Palgrave Macmillan.

Blume, A., 2003. Bertrand without fudge. Economics Letters 78, 167–168.

Dastidar, K.G., 2010. Existence of Bertrand equilibrium revisited. International Journal of Economic Theory (in press).

Hoernig, S.H., 2007. Bertrand games and sharing rules. Economic Theory 31, 573–585.

Saporiti, A., Coloma, G., 2010. Bertrand price competition in markets with fixed costs. B.E. Journal of Theoretical Economics 10 (1), (Contributions), Article 27.

Vives, X., 1999. Oligopoly Pricing: Old Ideas and New Tools. MIT Press, Cambridge, USA.