Game Theory with Application in Economics and Finance

Solution to the Final Exam, Magistère BFA 2, April 2024

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Duration: 90 mn. No document, no calculator allowed. Answers can be formulated in French or English.

Problem. Running Out of Bank Runs (15 pts)

Part A. Two depositors (6 pts)

A.1.a) (1 pt) From $n = 2$ and $r < \frac{1}{2}$, we have $2r < 1$. So, any withdrawal triggers a bank run. We then have $\bar{n}_W(r) = 0$. Player 1's corresponding payoff $g_1(x, y)$ associated to any pair of actions $(x, y) \in \{L, W\}^2$ writes as

$$g_1(L, L) = 1 + i;\ g_1(L, W) = 0;\ g_1(W, L) = 2r; \text{ and } g_1(W, W) = r.$$

A.1.b) (1 pt) These payoffs yield to player 1's best response correspondence:

— From $g_1(L, L) = 1 + i > 1 > 2r = g_1(W, L)$, we have $BR^1(L) = \{L\}$.

— From $g_1(L, W) = 0 < r = g_1(W, W)$, we have $BR^1(W) = \{W\}$.

By symmetry, we obtain the same best response correspondence for player 2. Hence, the set of pure-strategy Nash equilibrium is $Nash = \{(W, W); (L, L)\}$.

A.1.c) (1 pt) From the symmetry of players' payoffs and $g_1(L, L) = 1 + i > g_1(W, L) = 2r > g_1(W, W) = r > g_1(L, W) = 0$ we deduce that the set of Pareto efficient outcomes is the singleton $\{(L, L)\}$.

A.1.d) (1 pt) The corresponding payoff matrix writes as

$$\begin{pmatrix} 1\backslash 2 & W & L \\ W & r, r & 2r, 0 \\ L & 0, 2r & 1+i, 1+i \end{pmatrix}$$

A.2) (2 pts)

A.2.a) From $n = 2$ and $r \in [\frac{1}{2}; 1)$, we have $2 > 2r \geq 1$. So, two withdrawals are required to trigger a bank runk. We then have $\bar{n}_W(r) = 1$. Player 1's corresponding payoff $g_1(x, y)$ associated to any pair of actions $(x, y) \in \{L, W\}^2$ writes as

$$g_1(L, L) = 1 + i;\ g_1(L, W) = 1 + i;\ g_1(W, L) = 1; \text{ and } g_1(W, W) = r.$$

A.2.b) These payoffs yield to player 1's best response correspondence:

— From $g_1(L, L) = 1 + i > 1 = g_1(W, L)$, we have $BR^1(L) = \{L\}$.

— From $g_1(L, W) = 1 + i > r = g_1(W, W)$, we have $BR^1(W) = \{L\}$.

L is then player 1's strictly dominant strategy. By symmetry, we obtain the same best response correspondence for player 2. Hence, the set of pure-strategy Nash equilibrium is $Nash = \{(L, L)\}$.

A.2.c) From the symmetry of players' payoffs and $g_1(L, L) = g_1(L, W) = 1 + i > g_1(W, L) = 1 > g_1(W, W) = r$ we deduce that the set of Pareto efficient outcomes is still the singleton $\{(L, L)\}$.

A.2.d) The corresponding payoff matrix writes as

$$\begin{pmatrix} 1\backslash 2 & W & L \\ W & r, r & 1, 1+i \\ L & 1+i, 1 & 1+i, 1+i \end{pmatrix}$$

Part B. More than two depositors (5 pts)

Assume n depositors, with $n \geq 3$.

B.1) (1 pt) When $n_W(s_{-i}) < \bar{n}_W(r)$, there is no bank runk whatever player i's action. We then have $g_i(s_{-i}, W) = 1 < 1 + i = g_i(s_{-i}, L)$, so i's best response correspondence is worth $BR^i(s_{-i}) = \{L\}$.

B.2) (1 pt) When $n_W(s_{-i}) = \bar{n}_W(r)$, the bank runk occurrence depends on player i's action. We then have $g_i(s_{-i}, W) = \frac{nr}{n_W(s_{-i})+1} < 1 + i = g_i(s_{-i}, L)$, so i's best response correspondence is worth $BR^i(s_{-i}) = \{L\}$.

B.3) (1 pt) When $n_W(s_{-i}) > \bar{n}_W(r)$, there is a bank runk whatever player i's action. We then have $g_i(s_{-i}, W) = \frac{nr}{n_W(s_{-i})+1} > 0 = g_i(s_{-i}, L)$, so i's best response correspondence is worth $BR^i(s_{-i}) = \{W\}$.

B.4) (1 pt) From the previous answers, player i's best response consists in confirming the outcome obtained by other depositors' moves. When they do not trigger a bank run ($n_W(s_{-i}) \leq \bar{n}_W(r)$) player i chooses L, while when they trigger a bank run ($n_W(s_{-i}) > \bar{n}_W(r)$) player i chooses W. By symmetry, the maximal number of pure-strategy Nash equilibrium is two: either all depositors leave their money in the bank ($s = (L, L, ..., L)$) or all depositors run and withdraw ($s = (W, W, ...W)$).

B.5) (1 pt) From the previous answers, when $n_W(s_{-i}) \leq \bar{n}_W(r)$, L is player i's strictly dominant strategy. By symmetry, the same argument applies to all players. This makes a bank run incompatible with equilibrium behaviors. Such a condition translates into $\bar{n}_W(r) \geq n - 1$, so $r \geq \frac{n-1}{n}$.

Part C (4 pts). Incorporating deposit insurance

C.1) (2 pts) The corresponding payoff matrix writes as

$$\begin{pmatrix} 1\backslash 2 & W & L \\ W & \max\{I, r\}, \max\{I, r\} & \max\{I, 2r\}, I \\ L & I, \max\{I, 2r\} & 1+i, 1+i \end{pmatrix}$$

As in **A.1)**, the set of pure-strategy Nash equilibrium is $Nash = \{(W, W); (L, L)\}$ and the set of Pareto efficient outcomes is the singleton $\{(L, L)\}$.

C.2) (2 pts) The corresponding payoff matrix writes as

$$\begin{pmatrix} 1\backslash 2 & W & L \\ W & 1-\gamma, 1-\gamma & 1-\gamma, 1 \\ L & 1, 1-\gamma & 1+i, 1+i \end{pmatrix}$$

Now, L becomes a strictly dominant strategy. The set of pure-strategy Nash equilibrium is $Nash = \{(L, L)\}$ which corresponds to the set of Pareto efficient outcomes.

P.S.: This exercise is partially inspired from Libich, J., Nguyen, D. T., & Kiss, H. J. (2023). Running Out of Bank Runs. *Journal of Financial Services Research*, 64(1), 1-39.

Exercise. Infinitely repeated Prisoner's dilemma (5 pts)

1) (3 pts)

The grim trigger strategy here involves player i playing:

— c_i at period $t = 1$;

— then at period $t > 1$, playing c_i if (c_1, c_2) has been played until period $(t-1)$, and playing t_i otherwise.

When the game is repeated infinitely, the expected payoff along the cooperation path is written as

$$3 \times \sum_{t=0}^{+\infty} \delta^t = \frac{3}{1-\delta}$$

The highest expected payoff from deviation at period k is written as:

$$3 \times \sum_{t=0}^{k-1} \delta^t + (4+\alpha)\delta^{k+1} \times \sum_{t=k+1}^{+\infty} \delta^t$$
$$= \frac{3 \times (1-\delta^k) + (4+\alpha) \times (1-\delta)\delta^k + \delta^{k+1}}{1-\delta}.$$

The first expression is greater than the second if and only if

$$3 \times \delta^k \geq (4+\alpha) \times \delta^k + (1-(4+\alpha))\delta^{k+1}$$

That is, when

$$\delta^k(1+\alpha) \leq \delta^{k+1}(3+\alpha)$$

and thus

$$\delta \geq \frac{1+\alpha}{3+\alpha} \equiv \bar{\delta}(\alpha).$$

2) (2 pts) Clearly, $\frac{\partial \bar{\delta}(\alpha)}{\partial \alpha} = \frac{2}{(3+\alpha)^2} > 0$. Thus, the threshold$\bar{\delta}$ is increasing with α. This result corresponds to the intuition that the higher the unilateral deviation from mutual cooperation is profitable, i.e., the higher α is, the more players need to value the future (high δ) so that the prospect of future punishment encourages them not to betray the current cooperation.