Arbitrage and Pricing – Solution to the Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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Answers can be formulated in English or French.

Part A : Valuable strike of an American option (4 pts)

A.1) (3 pts) Any strike price K for which there is early exercise satisfies

$$S_0 - K \geq e^{-r}(q \max\{0; uS_0 - K\} + (1-q)\max\{0; dS_0 - K\}) \tag{1}$$

where q denotes the risk-neutral probability which is worth $q = \frac{e^r - d}{u-d}$.

Given that $K \geq dS_0$, there are two cases :

— (i) $K > uS_0$, so that $\max\{0; uS_0 - K\} = \max\{0; dS_0 - K\} = 0$.

— (ii) $uS_0 \geq K \geq dS_0$, so that $\max\{0; uS_0 - K\} \geq 0 = \max\{0; dS_0 - K\}$.

The first case leads to a contradiction. Indeed condition (1) then writes as $S_0 - K \geq 0$, which means that $uS_0 < K \leq S_0$, so $u < 1$ which contradicts $e^r < u$ with $r \geq 0$.

In the second case, condition (1) writes as $S_0 - K \geq e^{-r}q(uS_0 - K)$, which is equivalent to $K \leq S_0 \frac{1-e^{-r}qu}{1-e^{-r}q}$ (from $d < e^r < u$ we have $q < 1$, so $r \geq 0$ implies that $e^{-r}q < 1$). Hence

$$K \leq \min\{S_0 \frac{1 - e^{-r}qu}{1 - e^{-r}q}; uS_0\} := \bar{K}$$

A.2) (1 pt) The numerical application : $S_0 = 100$, $r = 2\%$, $u = 1.08$, $d = 0.96$ gives

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.02} - 0.96}{1.08 - 0.96} \simeq 50.17\%$$

and

$$K \leq \min\{S_0 \frac{1 - e^{-r}qu}{1 - e^{-r}q}; uS_0\} = \min\{100 \frac{1 - e^{-0.02} \times 0.5017 \times 1.08}{1 - e^{-0.02} \times 0.5017}; 108\} \simeq 92.26.$$

Hence, the maximal (rounded to the nearest euro) $\bar{K}$ is worth 92 euros.

Part B : A derivative in two steps binomial tree (9 pts)

B.1) (1 pt) The forward price for delivery of one share of the stock at time T is

$$F_{0,T}(S) = S_0 e^{rT}$$

B.2) (1 pt) The corresponding up and down moves u and d that depict the possible evolution of the underlying asset satisfy $S_u^1 = uS_0$ and $S_d^1 = dS_0$. That is

$$u = \frac{S_u^1}{S_0} = \frac{F_{0,T}(S)e^{\sigma\sqrt{T}}}{S_0} = e^{rT}e^{\sigma\sqrt{T}} = e^{rT+\sigma\sqrt{T}}$$

and

$$d = \frac{S_d^1}{S_0} = \frac{F_{0,T}(S)e^{-\sigma\sqrt{T}}}{S_0} = e^{rT}e^{-\sigma\sqrt{T}} = e^{rT-\sigma\sqrt{T}}$$

B.3) (1 pt) The assumption that needs to be made on σ so that the no-arbitrage condition with respect to u, d, and r holds for the binomial asset pricing model writes as

$$d < e^{rT} < u$$

which is equivalent

$$e^{rT-\sigma\sqrt{T}} < e^{rT} < e^{rT+\sigma\sqrt{T}}$$

that is

$$e^{-\sigma\sqrt{T}} < 1 < e^{\sigma\sqrt{T}}$$

Hence, the volatility must be strictly positive : $\sigma > 0$. This aligns with the notion of the stock being viewed as a risky asset.

B.4) (1 pt) The ratio S_u^1/S_d^1 writes as

$$\frac{S_u^1}{S_d^1} = \frac{F_{0,T}(S)e^{\sigma\sqrt{T}}}{F_{0,T}(S)e^{-\sigma\sqrt{T}}} = e^{2\sigma\sqrt{T}}.$$

The ratio $\frac{u}{d}$ has the same value.

B.5) (1 pt) The expression for the risk-neutral probability in the forward tree is

$$p = \frac{e^{rT} - d}{u - d}$$

Using the ratio $\frac{u}{d}$ from the previous answer, we have

$$p = \frac{\frac{e^{rT}}{d} - 1}{\frac{u}{d} - 1} = \frac{\frac{e^{rT}}{d} - 1}{e^{2\sigma\sqrt{T}} - 1} = \frac{e^{\sigma\sqrt{T}} - 1}{e^{2\sigma\sqrt{T}} - 1} = \frac{e^{\sigma\sqrt{T}} - 1}{(e^{\sigma\sqrt{T}} - 1)(e^{\sigma\sqrt{T}} + 1)} = \frac{1}{e^{\sigma\sqrt{T}} + 1}$$

Hence $x = e^{\sigma\sqrt{T}}$.

B.6) (1 pt) The limit of p as $T \to 0$ is $\frac{1}{2}$.

B.7) (3 pts) The price of a one-year, at-the-money European call option on this stock consistent with the above stock-price model is obtained by computing the corresponding values of u, d, p, S_1^u, S_1^d, C_1^u, and C_1^d. Given that the European call option is at-the-money, we deduce that $K = S_0$. Formally, applying the previous answers with $T = 1$, we obtain

$$C_0 = e^{-r}(pC_1^u + (1 - p)C_1^d)$$

with

$$C_1^u = \max\{0; S_1^u - K\} = \max\{0; (u-1)S_0\} = \max\{0; (e^{r+\sigma} - 1)S_0\} = \max\{0; (e^{0.29} - 1)70\} \simeq 23.55$$

$$C_1^d = \max\{0; S_1^d - K\} = \max\{0; (d-1)S_0\} = \max\{0; (e^{r-\sigma} - 1)S_0\} = \max\{0; (e^{-0.21} - 1)70\} = 0$$

and

$$p = \frac{1}{e^{\sigma} + 1} = \frac{1}{e^{0.25} + 1} \simeq 43.78\%.$$

Therefore

$$C_0 = e^{-0.04}(0.4378 \times 23.55) \simeq 9.91$$

Part C : Forward Contract Pricing and Greeks (5 pts)

C.1) (1 pt) The put-call parity equation at time t writes as

$$C_t + Ke^{-r(T-t)} = P_t + S_t$$

C.2) (1 pt) From the previous equation, differentiating with respect to S, we obtain

$$\Delta(C) = \Delta(P) + 1$$

C.3) (1 pt) Differentiating with respect to S again, we obtain

$$\Gamma(C) = \Gamma(P)$$

C.4) (1 pt) Differentiating the put-call parity with respect to σ, we obtain

$$\nu(C) = \nu(P)$$

C.5) (1 pt) Differentiating the put-call parity with respect to t, we obtain

$$\Theta(C) + rKe^{-r(T-t)} = \Theta(P)$$

Part D : QCM (2 pts)

D.1) (1 pt) (a) The call-option price will drop. Dividends have the effect of reducing the stock price on the ex-post dividend date.

D.2) (1 pt) (c) Call on a non-dividend-paying stock. (see Chapter 9.)