Game Theory with Application in Economics and Finance

Solution to the Final Exam, Magistère BFA 2, March 2023

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Duration : 90 mn. No document, no calculator allowed. Answers can be formulated in French or English.

The financial trilemma

Part A (8 pts). Two countries.

Assume $N := \{1, 2\}$.

A.1)

A.1.a) (1 pt) From [H1] we have $\alpha_i B - C < 0$, so $\neg P$ is country i's strictly dominant strategy (i.e., $\neg P \succ_i P$). There is then a unique Nash equilibrium. It consists for the two countries not to participate to the bailout : $Nash = \{(\neg P, \neg P)\}$.

A.1.b) (1 pt) The set of Pareto efficient outcomes writes as : $P = \{(P, \neg P); (\neg P, P); (P, P)\}$. Indeed,

- the outcome $(\neg P, P)$ (resp. $(P, \neg P)$) is the only outcome that provides country 1 (resp. 2) with his maximal payoff;
- From [H1] we have $\alpha_i B - \frac{C}{2} > 0$, so the outcome $(\neg P, \neg P)$ is Pareto dominated by the outcome (P, P); and
- (P, P) by which the cost of the bailout is shared among the two countries is neither Pareto dominated by $(P, \neg P)$ or $(\neg P, P)$ where one of the two countries has to finance the entire bailout solely.

A.1.c) (1 pt) Draw the corresponding payoff matrix and report your previous answers using arrows and the symbols (N) and (P) to indicate the outcomes that are Nash equilibrium and/or Pareto efficient.

The corresponding payoff matrix writes as

1\2	P	$\neg P$
P	$\alpha_1 B - \frac{C}{2}; \alpha_2 B - \frac{C}{2}$ (P)	$\alpha_1 B - C; \alpha_2 B$ (P)
$\neg P$	$\alpha_1 B; \alpha_2 B - C$ (P)	$0; 0$ (N)

A.1.d) (1 pt) This game is a prisoner's dilemma. Indeed, the jointly preferred outcome (P, P) requires both countries to play their strictly dominated strategy P.

A.1.e) (1 pt) No, in a case of sequential interaction the previous results would not change because each country would play his strictly dominant strategy anyway.

A.2)

A.2.a) (1 pt) From [H2] we have $0 > \alpha_F B - C$, so $\neg P$ is country F's strictly dominant strategy (i.e., $\neg P \succ_F P$). From [H2] we have $\alpha_H B - C > 0$, so country H's best response to $\neg P$ is P. There is then a unique Nash equilibrium. It consists for the home country to bailout the entire financial institution solely : $Nash = \{(P, \neg P)\}$.

A.2.b) (1 pt) The set of Pareto efficient outcomes writes as : $P = \{(P, \neg P); (\neg P, P); (P, P)\}$. Indeed,

— the outcome $(\neg P, P)$ (resp. $(P, \neg P)$) is the only outcome that provides country H (resp. F) with his maximal payoff ;

— From [H2] we have $\alpha_H B - C > 0$, so the outcome $(\neg P, \neg P)$ is Pareto dominated by the outcome $(P, \neg P)$; and

— (P, P) by which the cost of the bailout is shared among the two countries is neither Pareto dominated by $(P, \neg P)$ or $(\neg P, P)$ where one of the two countries would have to finance the entire bailout solely.

A.2.c) (1 pt) The corresponding payoff matrix writes as (with $1 = H$ and $2 = F$) :

$1\backslash 2$	P	$\neg P$
P	$\alpha_1 B - \frac{C}{2}; \alpha_2 B - \frac{C}{2}$	$\alpha_1 B - C; \alpha_2 B$
$\neg P$	$\alpha_1 B; \alpha_2 B - C$	$0; 0$

Part B (8 pts). More than two countries.

Assume n countries, with $n \geq 3$.

B.1) (1 pt) The payoffs are $g_i((s_i = \neg P, s_{-i})) = 0$ and $g_i((s_i = P, s_{-i})) = \alpha_i B - C$, so the condition writes as $\alpha_i B - C < 0$.

B.2) (1 pt) We have $g_i((s_i = \neg P, s_{-i})) = \alpha_i B > \alpha_i B - \frac{C}{k+1} = g_i((s_i = P, s_{-i}))$.

B.3) (1 pt) Yes, from **B.1)** and **B.2)** country i has a strictly dominant strategy which consists in not participating to the bailout (i.e., $\neg P \succ_i P$).

B.4) (1 pt) From **B.3)** there is a unique Nash equilibrium. It consists for all countries not to participate to the bailout : $Nash = \{(\neg P, \neg P, ..., \neg P)\}$.

B.5) (1 pt) No. From [H4], we have $0 > \alpha_i B - C$ for any $i \in N$, so the hypothesis [H3] holds. From **B.4)** the set of Nash equilibrium is then the singleton $\{(\neg P, \neg P, ..., \neg P)\}$. From [H4], we also have $\alpha_i B - \frac{C}{n} > 0$, so $g_i(P, P, ..., P) > 0 = g_i(\neg P, \neg P, ..., \neg P)$. Hence, the unique Nash equilibrium $(\neg P, \neg P, ..., \neg P)$ is Pareto-dominated by $(P, P, ..., P)$, so it is not efficient.

B.6) (1 pt) There are three cases to consider : i) $s_{-i} \neq (\neg P, \neg P, ..., \neg P)$ or $\alpha_i B - C < 0$; ii) $s_{-i} = (\neg P, \neg P, ..., \neg P)$ and $\alpha_i B - C > 0$; and iii) $s_{-i} = (\neg P, \neg P, ..., \neg P)$ and $\alpha_i B - C = 0$.

First, consider case i). When $s_{-i} \neq (\neg P, \neg P, ..., \neg P)$, there is at least one contributor among the counterparts of country i. From **B.2)** $\neg P$ is country i's best-response. When $\alpha_i B - C < 0$, from **B.3)** $\neg P$ is country i's strictly dominant strategy. In both situations, $\neg P$ is country i's best-response.

Second, consider case ii). From $\alpha_i B - C > 0$ and $s_{-i} = (\neg P, \neg P, ..., \neg P)$, we have $g_i((s_i = \neg P, s_{-i})) = 0 < \alpha_i B - C = g_i((s_i = P, s_{-i}))$. So, P is country i's best-response.

Third, consider case iii). From $\alpha_i B - C = 0$ and $s_{-i} = (\neg P, \neg P, ..., \neg P)$, we have $g_i((s_i = \neg P, s_{-i})) = 0 = \alpha_i B - C = g_i((s_i = P, s_{-i}))$. So, both P and $\neg P$ are country i's best-response.

Therefore, country i's best-reponse correspondance in pure strategies writes as :

$$s_i^*(s_{-i}) = \begin{cases} \{\neg P\} & \text{if} \quad \text{i) } s_{-i} \neq (\neg P, \neg P, ..., \neg P) \text{ or } \alpha_i B - C < 0 \\ \{P\} & \text{if} \quad \text{ii) } s_{-i} = (\neg P, \neg P, ..., \neg P) \text{ and } \alpha_i B - C > 0 \\ \{P, \neg P\} & \text{if} \quad \text{iii) } s_{-i} = (\neg P, \neg P, ..., \neg P) \text{ and } \alpha_i B - C = 0 \end{cases}$$

B.7) (1 pt) The bank is possibly rescued at equilibrium if and only if there is at least one country $i \in N$, and a profile of action of country i's counterparts s_{-i}, for which $P \in s_i^*(s_{-i})$. From **B.6)**, the condition writes as $s_{-i} = (\neg P, \neg P, ..., \neg P)$ and $\alpha_i B - C \geq 0$.

B.8) (1 pt) Clearly, there is a country i satisfying the condition $\alpha_i B - C \geq 0$ if and only if $\alpha_H B - C \geq 0$. This rewrites as $\alpha_H \geq \frac{C}{B}$ and has the interpretation that the bank is not too financially integrated. So, from **B.7)** the bank is possibly rescued at equilibrium if and only if the home country is ready to refinance the entire institution solely because the national benefit of the bailout is high enough. We then obtain the financial trilemma in the sense that in the context of strategic national financial policies, the financial stability (by which ailing financial institutions are rescued) requires low financial integration (i.e., $1 - \alpha_H < \frac{C}{B}$) .

Indefinitely repeated Prisoner's Dilemma (4 pts)

(1 pt) From the one-shot deviation principle (which holds in any infinite horizon game where the discount factor is less than 1), (tit-for-tat,tit-for-tat) is a subgame perfect equilibrium if and only if there exist no profitable one-shot deviations for each subgame and every player.

(2 pts) Suppose that player 2 adheres to tit-for-tat. Consider player 1's behavior in subgames following histories that end in each of the following outcomes.

— (C, C) If player 1 adheres to tit-for-tat the outcome is (C, C) in every period, so that her discounted average payoff in the subgame is:

$$g_1((C,C),(C,C),...) = x \sum_{t=0}^{+\infty} \delta^t = \frac{x}{1-\delta}$$

If she chooses D in the first period of the subgame, then adheres to tit-for-tat, the outcome alternates between (D, C) and (C, D), and her discounted average payoff is:

$$\begin{aligned} g_1((D,C),(C,D),(D,C),(C,D),...) &= y + \delta^2 y + \delta^4 y + ... = y \sum_{t=0}^{+\infty} \delta^{2t} \\ &= \frac{y}{1-\delta^2} \end{aligned}$$

Such a one-period unilateral deviation from tit-for-tat is not profitable for player 1 if and only if

$$\frac{x}{1-\delta} \geq \frac{y}{1-\delta^2} \iff x \geq \frac{y}{1+\delta} \tag{1}$$

— (C, D) If player 1 adheres to tit-for-tat the outcome alternates between (D, C) and (C, D), so that her discounted average payoff is $g_1((D,C),(C,D),(D,C),(C,D),...)$. If she deviates to C in the first period of the subgame, then adheres to tit-for-tat, the outcome is (C, C) in every period, and her discounted average payoff is $g_1((C,C),(C,C),...)$. So the inequality (1) is reversed to obtain:

$$x \leq \frac{y}{1+\delta} \tag{2}$$

— (D, C) If player 1 adheres to tit-for-tat the outcome alternates between (C, D) and (D, C), so that her discounted average payoff is:

$$\begin{aligned} g_1((D,C),(C,D),(D,C),(C,D),...) &= \delta y + \delta^3 y + \delta^5 y + ... \\ &= y \sum_{t=0}^{+\infty} \delta^{2t+1} = \frac{\delta y}{1-\delta^2} \end{aligned}$$

If she deviates to D in the first period of the subgame, then adheres to tit-for-tat, the outcome is (D, D) in every period, and her discounted average payoff is:

$$g_1((D,D),(D,D),...) = \sum_{t=0}^{+\infty} \delta^t = \frac{1}{1-\delta}$$

Such a one-period unilateral deviation from tit-for-tat is not profitable for player 1 if and only if

$$\frac{\delta y}{1-\delta^2} \geq \frac{1}{1-\delta} \iff \delta y \geq 1+\delta \tag{3}$$

— (D, D) If player 1 adheres to tit-for-tat the outcome is (D, D) in every period, so that her discounted average payoff is $g_1((D,D),(D,D),...)$. If she deviates to C in the first period of the subgame, then adheres to tit-for-tat, the outcome alternates between (C, D) and (D, C), and her discounted average payoff is $g_1((D,C),(C,D),(D,C),(C,D),...)$. So the inequality (3) is reversed to obtain:

$$\delta y \leq 1+\delta \tag{4}$$

(1 pt) Therefore, from equations (1)-(4), we obtain the system:

$$\begin{cases} x = \frac{y}{1+\delta} \\ \delta y = 1+\delta \end{cases} \iff \begin{cases} x = \frac{(1+\delta)/\delta}{1+\delta} = \frac{1}{\delta} \\ y - x = y - \frac{y}{1+\delta} = \frac{\delta y}{1+\delta} = \frac{\delta}{1+\delta}\frac{1+\delta}{\delta} = 1 \end{cases} \iff \begin{cases} \delta = \frac{1}{x} \\ y - x = 1 \end{cases}$$

The same arguments apply to deviations by player 2, so we conclude that (tit-for-tat, tit-for-tat) is a subgame perfect equilibrium if and only if $y - x = 1$ and $\delta = \frac{1}{x}$.