Arbitrage&Pricing

Additional Exercises from Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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Part A : Two steps binomial tree (5 pts)

There are three periods, $t \in \{0, 1, 2\}$. There are two assets. One non-risky asset (money that can be borrowed or lend) that returns 5% per period with continuous compounding. And a risky asset which is a commodity whose price starts at $S_0 = 50$€ at time 0. At each date $t > 0$, there is either an upward or a downward move. The price of the commodity is then either multiplied by $u = 1.1$ or $d = 0.9$. We consider two European options : a call and a put on the commodity with the same strike $K = 55$€ and maturity $T = 2$.

A.1) (1 pt) Draw the binomial tree that depicts the evolution of the stock price through time t.

A.2) (1 pts) Compute the call price C_t at each date $t \in \{0, 1, 2\}$ and report its value on the binomial tree.

A.3) (1 pts) Compute the put price P_t at each date $t \in \{0, 1, 2\}$ and report its value on the binomial tree.

A.4) (2 pts) Assume we are at date $t = 1$ and there was an upward move before (so the risky asset is worth S_1^u). Suppose the market value of the put is $P^u_{1,market} = 1,26$. Construct an arbitrage that uses one unit of the option.

Part B : A *chooser option* in two steps binomial tree (3 pts)

A ***chooser option*** is an option contract that allows the holder to decide whether it is to be a call or put prior to the expiration date. We consider a *chooser option* that consists for the purchaser to choose at date $t = 1$ whether his option is the European call or the European put described in the previous part. Let Π_t denote the *chooser option* price through time $t \in \{0, 1, 2\}$.

B.1) (1 pt) What decision rule would a rational investor follow to convert his *chooser option* at date $t = 1$?

B.2) (2 pts) Draw the binomial tree that depicts the evolution of the *chooser option* price through time $t \in \{0, 1, 2\}$, denoted as Π_t. Is the tree recombining ? Why ?

Part C : A *chooser option* in continuous time (8 pts)

We consider now a time interval of the form $[0; T]$. The risk-free interest rate is r% and is continuously compounded. Let C_t and P_t, with $t \in [0; T]$, denote the respective prices of a European call and a European put on the same underlying asset whose price is S_t, with same strike K and maturity T. Let Π_t^τ denote the *chooser option* price at date t that consists for the purchaser to choose at a given date $\tau \in (0; T)$ whether his option is the prevous European call or the European put. Let $\mathbb{Q}$ denote the risk neutral probability, I_t denote the information known at date t, and $\mathbb{E}^{\mathbb{Q}}[.|I_t]$ denote the expectation operator given information I_t.

C.1) (1 pt) What is the price $\Pi_{t=\tau}^{\tau}$ of the *chooser option* at date τ ? (Hint : Use the indicator function $\mathbb{1}_A$ that indicates whether event A happens.)

C.2) (1 pt) What is the price Π_t^{τ} of the *chooser option* at any date $t > \tau$?

C.3) (1 pt) Give the price Π_t^{τ} of the *chooser option* at any date $t < \tau$ as a function of times C_τ and P_τ.

C.4) (1 pt) Give the price Π_t^{τ} of the *chooser option* at any date $t < \tau$ as a function of times C_T and P_T.

C.5) (1 pt) Show that at any date $t < \tau$,the price Π_t^{τ} can be written as

$$\mathbb{E}^{\mathbb{Q}}[C_T - (C_T - P_T)\mathbb{1}_{\{C_\tau \leq P_\tau\}}|I_t]e^{-r(T-t)}$$

C.6) (1 pt) Show that the previous price Π_t^{τ} can be rewritten as

$$C_t + \mathbb{E}^{\mathbb{Q}}[(K - S_T)\mathbb{1}_{\{C_\tau \leq P_\tau\}}|I_t]e^{-r(T-t)}$$

C.7) (1 pt) From the previous expression, deduce the *chooser option* price Π_0 at date 0.

C.8) (1 pt) Show that the previous price Π_0^{τ} can be rewritten as

$$C_0 + \mathbb{E}^{\mathbb{Q}}[(Ke^{-r(T-\tau)} - S_\tau)^{+}e^{-r\tau}]$$

Part D : An equivalent portfolio (4 pts)

Consider a time interval of the form $[0;T]$ and a specific date $\tau \in (0;T)$. Let Π_t' denote a portfolio value, at date $t \in [0;T]$, consisting in a long position in a European call option with strike K and expiration date T and a long position in a European put option with strike $Ke^{-r(T-\tau)}$ and expiration date τ on the same underlying asset whose price is S_t. Let $\mathbb{Q}$ denote the risk neutral probability, and $\mathbb{E}^{\mathbb{Q}}[.]$ denote the expectation operator under probability $\mathbb{Q}$.

D.1) (1 pt) Give the portfolio value Π_0' at date 0.

D.2) (1 pt) What can you conclude with respect to the *chooser option* described in the previous part ?

D.3) (2 pts) Let $\mathcal{N}(.)$ denote the standard normal cumulative distribution function. What is the portfolio value Π_0' at date 0 according to the Black-Scholes formula ?