Arbitrage&Pricing

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Chapter 4

Chapter 4: Black–Scholes model
Outline

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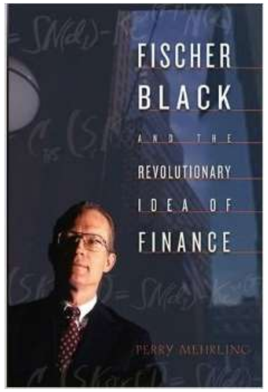
Introduction

- In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options.
- The Black–Scholes model was first published by Fischer Black and Myron Scholes in their 1973 paper
  - F. Black and M. Scholes, “The Pricing of Options and Corporate Liabilities,” Journal of Political Economy, 81, 1973: 637-59.
- Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model, and coined the term "Black–Scholes options pricing model".
  - R.C. Merton, “Theory of Rational Option Pricing,” Bell Journal of Economics and Management Science, 4, 1973: 141-83.
- The model has had a huge influence on the way that traders price and hedge derivatives.

Introduction



Merton and Scholes received the 1997 Nobel Prize in Economics (Black died in 1995).

Introduction

- The aim of this chapter is to introduce the Black-Scholes formula
  - It is an expression for the current value of a European call option on a stock (which pays no dividends before expiration), in a context of instantaneous aribtrage (delta hedging with abritarily small lenght of time).
- We will adopt an heurictic approach by deducing the formula from the results obtained in the previous chapters.
  - We will consider the same setup as in these chapters, with same assumption, except that the lenght of time for the arbitrage (delta hedging) will be considered as to be abritarily small.

Introduction

- Black and Scholes used the capital asset pricing model (CAPM) to determine a relationship between the market's required return on the option to the required return on the stock.
  - This was not easy because the relationship depends on both the stock price and time.
- Merton's approach was more general than that of Black and Scholes because it did not rely on the assumptions of the CAPM.
  - It involved setting up a riskless portfolio consisting of the option and the underlying stock and arguing that the return on the portfolio over a short period of time must be the risk-free return.
  - It derives the Black-Scholes-Merton model from a binomial tree by valuing a European option on a non- dividend-paying stock and allowing the number of time steps in the binomial tree to approach infinity.
  - The proof is relegated to the Appendix.

Chapter 4: Black–Scholes model
Outline

Heuristic Approach
Coming Back to the One-Period Binomial Model

- Consider a call option on a stock, with initial price S_0, with exercise price K, and maturing at time 1.
- In Chapter 2 (Proposition 2.5) we saw that the absence of arbitrage opportunities (NAO) implies that the current value of the call must be

$$C_0 = \frac{\mathbb{E}^{\mathbb{Q}}[C_1]}{1+r} = \frac{qC_1^u + (1-q)\,C_1^d}{1+r}$$

where $\mathbb{Q}$ such that $q = \frac{1+r-d}{u-d}$ is the equivalent martingale measure.

- When $d < r < u$ the option will be exercised if the stock price goes up ($S_1 = uS_0$), and it will expire worthless if the stock price goes down ($S_1 = dS_0$).

Heuristic Approach
Coming Back to the One-Period Binomial Model

- So, $C_1^u = uS_0 - K$ and $C_1^d = 0$. We can rewrite the previous call price formula like this:

$$C_0 = \frac{q\,(uS_0 - K)}{1+r} = \left(\frac{qu}{1+r}\right) S_0 - q\frac{K}{1+r}$$

- The factor $\frac{qu}{1+r}$ equals the factor by which the discounted expected value of contingent receipt of the stock exceeds the current value of the stock.
- Reasonning with continuous coupounding we would have

$$C_0 = \left(e^{-r}qu\right) S_0 - qe^{-r}K$$

Heuristic Approach
Coming Back to the n-Period Binomial Model

- Consider now that the call option matures at time T for which there are n intermediate market valuations of the stock.
- In Chapter 3 we saw that the absence of arbitrage opportunities (NAO) implies that the current value of the call must be

$$C_0 = \frac{\mathbb{E}^{\mathbb{Q}}[C_T]}{(1+r)^n} = \sum_{k=0}^{n} \frac{\binom{n}{k} q^k\,(1-q)^{n-k}\,C_T^{u^k d^{n-k}}}{(1+r)^n}$$

where $\mathbb{Q}$ such that $q = \frac{1+r-d}{u-d}$ is the equivalent martingale measure.

Heuristic Approach
Coming Back to the n-Period Binomial Model

- Reasonning with continuous coupounding we would have

$$C_0 = e^{-rT}\sum_{k=0}^{n}\binom{n}{k} q^k\,(1-q)^{n-k}\,C_T^{u^k d^{n-k}}.$$

- When $d < r < u$ there is a minimum number of upward moves necessary for the option to be exercised.
  - We say that the option is in the money.
  - Let a denotes this minimum number.
  - The option will then be exercised if the stock price goes up by at least a times ($S_T \geq u^a d^{n-a} S_0$), and it will expire worthless if the stock price goes down by less than a times($S_T < u^a d^{n-a} S_0$).

Heuristic Approach
Coming Back to the n-Period Binomial Model

- So, $C_T^{u^k d^{n-k}} = \begin{cases} u^k d^{n-k} S_0 - K \text{ if } k \geq a \\ 0 \text{ otherwise} \end{cases}$.
- Hence, we can rewrite the previous call price formula like this:

$$\begin{aligned} C_0 &= e^{-rT} \sum_{k=a}^{n} \binom{n}{k} q^k (1-q)^{n-k} \left(u^k d^{n-k} S_0 - K \right). \\ &= \left(e^{-rT} \sum_{k=a}^{n} \frac{n!}{k!\,(n-k)!} (qu)^k (d(1-q))^{n-k} \right) S_0 \\ &\quad - \sum_{k=a}^{n} \frac{n!}{k!\,(n-k)!} q^k (1-q)^{n-k} e^{-rT} K \end{aligned}$$

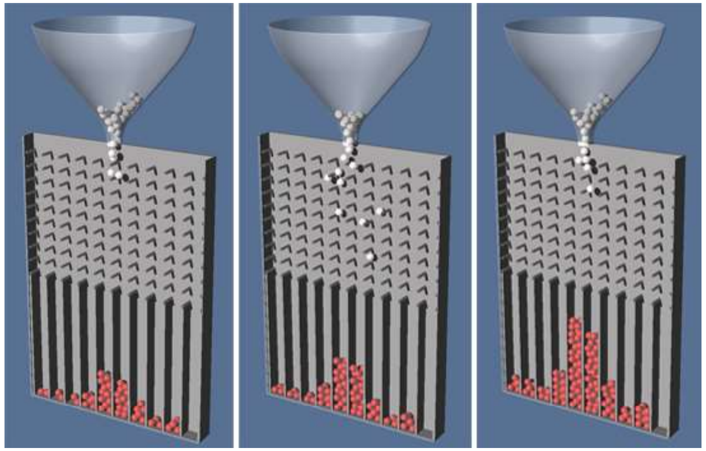
Heuristic Approach
Volatility Erodes Return

- Consider $S_0 = 100$, $u = 1.1$, and $d = 0.9$.
  - What is the value of S_2^{ud}?
  - What is the value of $S_4^{u^2 d^2}$?
- Compute now the same values for $u = 1.3$, and $d = 0.7$.
  - What do you obtain?
  - Conclude.

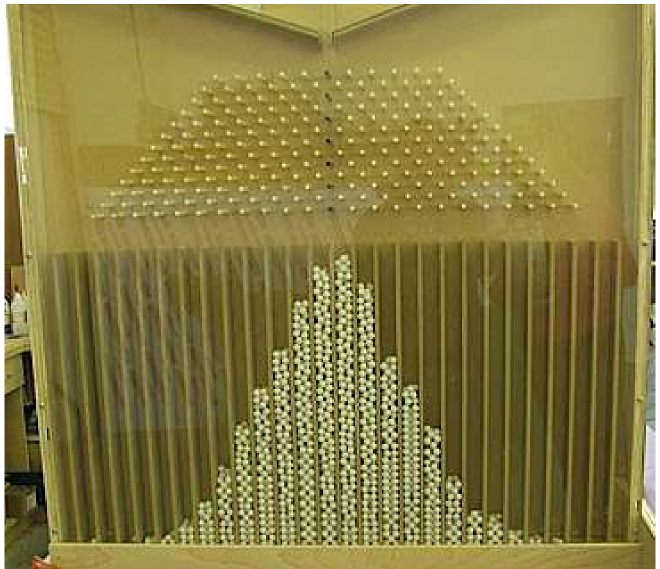
Property

Volatility erodes returns.

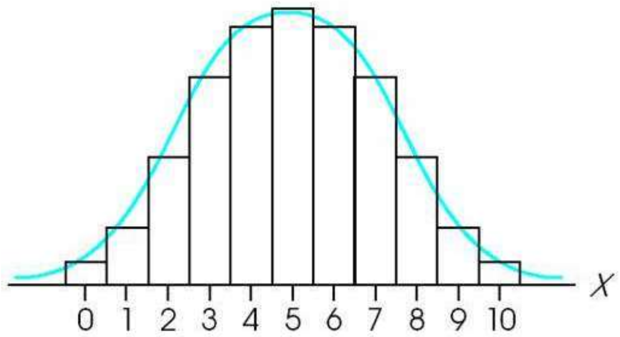
Heuristic Approach
From Binomial to Normal distribution



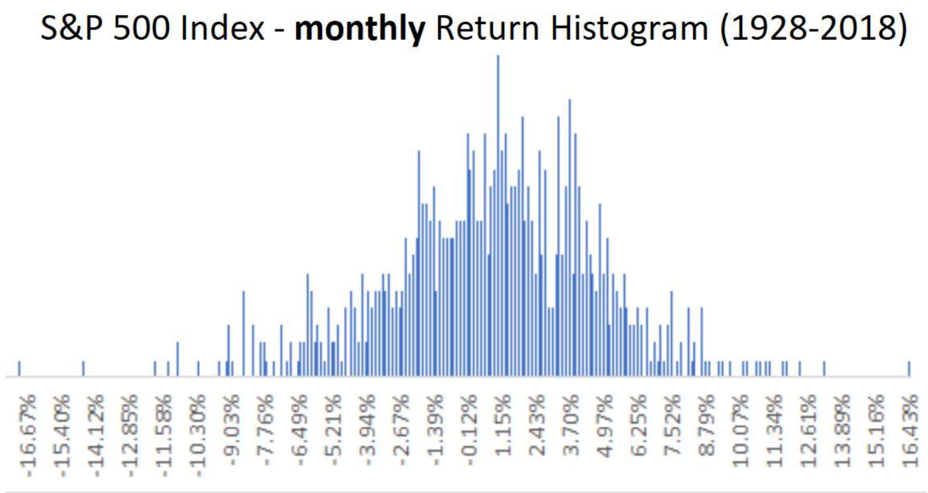
Heuristic Approach
From Binomial to Normal distribution



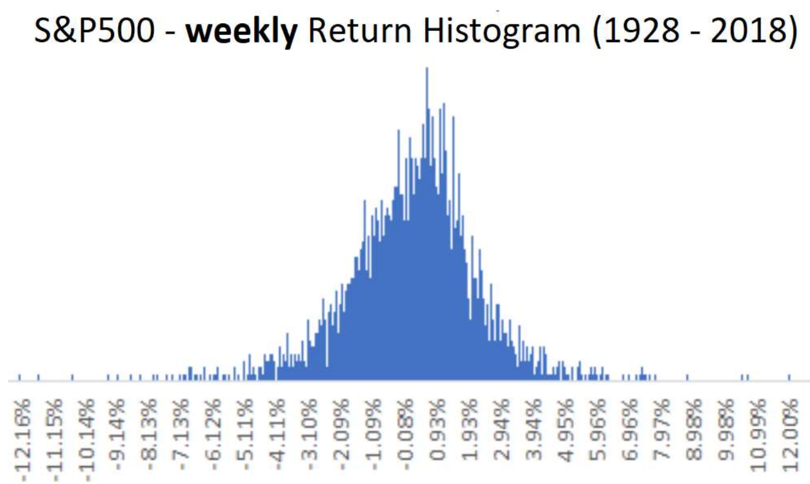
Heuristic Approach
From Binomial to Normal distribution



Heuristic Approach
From Binomial to Normal distribution



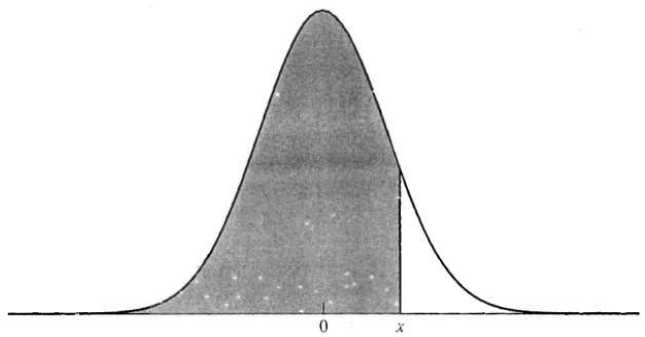
Heuristic Approach
From Binomial to Normal distribution



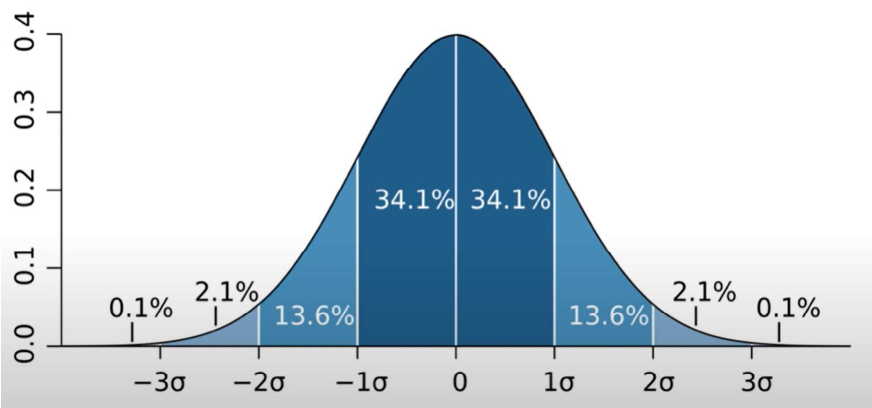
Heuristic Approach

- In the following, we will use the cumulative probability distribution function for a standardized normal distribution, denoted as $\mathcal{N}(x)$
- $\mathcal{N}(x)$ is then the probability that a variable with a standard normal distribution will be less than x.

Shaded area represents $N(x)$.



Heuristic Approach
From Binomial to Normal distribution



Heuristic Approach
From Binomial to Normal distribution

- Taking into account that volatility erodes returns, we will use that if the log return of a stock price is normally distributed then its mean is not r but is instead $\left(r - \frac{1}{2}\sigma^2\right)$.
  - Said differently, for a normal distribution, volatility erodes returns about half the variance.

Heuristic Approach
Log return normally distributed

- Black-Scholes assume that the stock price at time T (and any subsequent time t, replacing T with t) is

$$S_T = S_0 R(T)$$

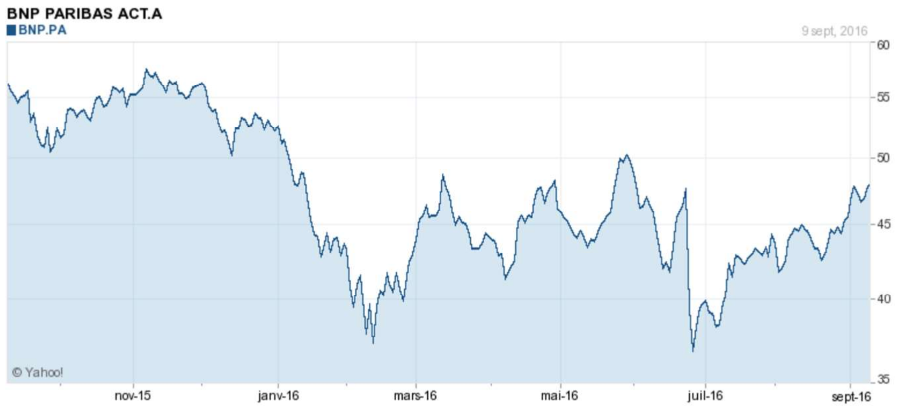
where the log return of the stock price is normally distributed under the equivalent martingale measure $\mathbb{Q}$, with mean $\left(r - \frac{1}{2}\sigma^2\right) T$ and variance $\sigma^2 T$:

$$\ln R(T) \overset{\mathbb{Q}}{\sim} N\left(\left(r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right)$$

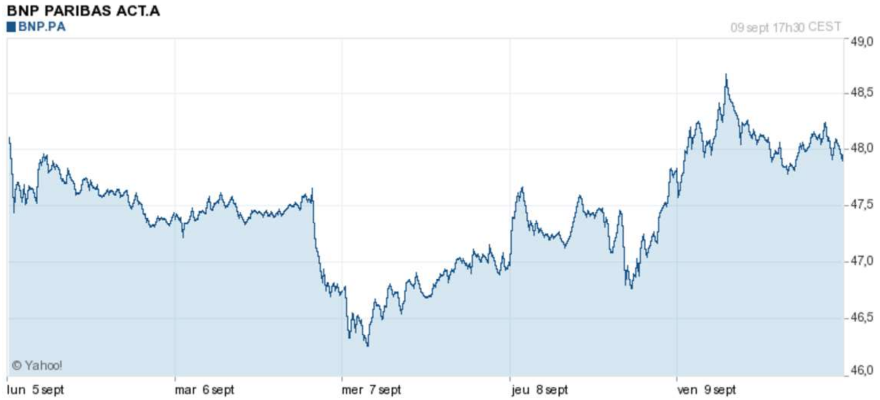
- It means that:
  - the expected returns are independent of the stock price;
  - the stock price only takes positive values;
  - the stock price is continuous everywhere but differentiable nowhere.

Heuristic Approach
Log return normally distributed

Heuristic Approach
Log return normally distributed

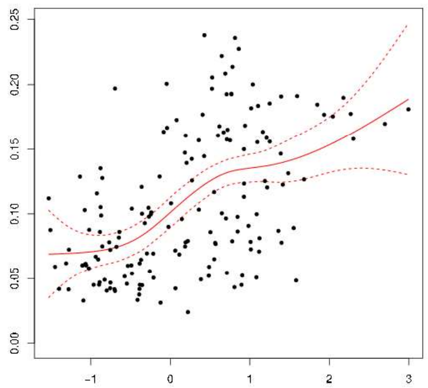


Heuristic Approach
Log return normally distributed



Heuristic Approach
Log return normally distributed

- In an attempt to make the model for stock prices more realistic, some papers drop the assumption that the volatility is constant.
  - A model that assumes that the volatility is a deterministic function of the stock price and time is called *local volatility model*.



Heuristic Approach
Black-Scholes Formula

- In this setup, the absence of arbitrage opportunities (NAO) implies that the current value of the call must be

$$C_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[C_T]$$

- From

$$C_T = (S_T - K)^+ = \begin{cases} S_T - K \text{ if } S_T > K \\ 0 \text{ otherwise} \end{cases}$$

we have

$$\begin{aligned} C_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T - K | S_T > K] \\ &= e^{-rT}\left(\mathbb{E}^{\mathbb{Q}}[S_T | S_T > K] - \mathbb{E}^{\mathbb{Q}}[K | S_T > K]\right) \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T | S_T > K] - e^{-rT}K\mathbb{Q}[S_T > K] \end{aligned}$$

Heuristic Approach
Black-Scholes Formula

- Let us first compute $\mathbb{Q}[S_T > K]$.
  - From $S_T = S_0 R(T)$ we have
$$\ln S_T = \ln S_0 + \ln R(T).$$
  - From
$$\ln R(T) \overset{\mathbb{Q}}{\sim} N\left(\left(r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right)$$
    we have
$$\ln S_T \overset{\mathbb{Q}}{\sim} N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right)$$
  - So,
$$\frac{\ln S_T - \mathbb{E}[\ln S_T]}{\sqrt{\mathbb{V}[\ln S_T]}} \overset{\mathbb{Q}}{\sim} N(0,1)$$
  - That is
$$\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T\right)}{\sqrt{\sigma^2 T}} \overset{\mathbb{Q}}{\sim} N(0,1)\,.$$

Heuristic Approach
Black-Scholes Formula

- Now, let us denote $\mathcal{N}(x) := \mathbb{P}[X \leq x]$ when $X \overset{\mathbb{P}}{\sim} N(0,1)$.
  - So we have
$$\mathbb{P}[X > x] = 1 - \mathcal{N}(x) = \mathbb{P}[X < -x] = \mathcal{N}(-x)$$
  - Using that $S_T > K$ is equivalent to
$$\frac{\ln S_T - \left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T\right)}{\sqrt{\sigma^2 T}} > \frac{\ln K - \left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T\right)}{\sqrt{\sigma^2 T}}$$
  - We obtain
$$\mathbb{Q}[S_T > K] = \mathcal{N}\left(-\frac{\ln K - \left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T\right)}{\sqrt{\sigma^2 T}}\right)$$
  - That is
$$\mathbb{Q}[S_T > K] = \mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)$$

Heuristic Approach
Black-Scholes Formula

- Now let us compute $\mathbb{E}^{\mathbb{Q}}[S_T | S_T > K]$.

Property

If $\ln X \sim N(\mu, s^2)$ *then* $\mathbb{E}[X | X > K] = e^{\mu + \frac{s^2}{2}} \mathcal{N}\left(\frac{\mu + s^2 - \ln K}{s}\right)$.

- Using this Property, with
$$\ln S_T \overset{\mathbb{Q}}{\sim} N\left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T, \sigma^2 T\right)$$
  we have $e^{\mu + \frac{s^2}{2}} = e^{\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T + \frac{\sigma^2 T}{2}} = e^{\ln S_0 + rT} = S_0 e^{rT}$ and
$$\frac{\mu + s^2 - \ln K}{s} = \frac{\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right) T + \sigma^2 T - \ln K}{\sqrt{\sigma^2 T}} = \frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sqrt{\sigma^2 T}}$$

Heuristic Approach
Black-Scholes Formula

- Hence, according to the Property we have
$$\mathbb{E}^{\mathbb{Q}}[S_T | S_T > K] = S_0 e^{rT} \mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sqrt{\sigma^2 T}}\right)$$
- Therefore
$$C_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}}[S_T | S_T > K] - e^{-rT} K \mathbb{Q}[S_T > K].$$
  with
$$e^{-rT} \mathbb{E}^{\mathbb{Q}}[S_T | S_T > K] = S_0 \mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sqrt{\sigma^2 T}}\right)$$
  and
$$\mathbb{Q}[S_T > K] = \mathcal{N}\left(\frac{\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sqrt{\sigma^2 T}}\right)$$

Heuristic Approach
Black-Scholes Formula

- We are now able to state the Black-Scholes Formula.

Theorem (Black-Scholes-Merton Formula for Call Option)

The price of European call, C_0, write as

$$C_0 = S_0 \mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

where

$$d_1 \equiv \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

and

$$d_2 \equiv \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Heuristic Approach
Black-Scholes Formula

- Similarly, with the same d_1 and d_2 of the previous Theorem, we have obtain the price of a Put with similar characteristics.

Corollary (Black-Scholes-Merton Formula for Put Option)

The price of European put, P_0, write as

$$P_0 = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$

Chapter 4: Black–Scholes model
Outline

Example

Example

The stock price 6 months from the expiration of an European option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum.

What are the values of the European call and put?

Example

Solution

Example

Solution

Example

Solution

Chapter 4: Black–Scholes model
Outline

Conclusion

- According to the Black-Scholes Formula, the prices of European call, C_0, and European put, P_0, on a non dividend paying stock with initial price S_0, and volatility σ, with a strike K and maturity T, when the risk-free interest rate is r write as
$$C_0 = S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$
and
$$P_0 = Ke^{-rT}\mathcal{N}(-d_2) - S_0\mathcal{N}(-d_1)$$
where $\mathcal{N}(\cdot)$ denotes the cumulative probability distribution function for a standardized normal distribution,
$$d_1 \equiv \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
and
$$d_2 \equiv \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Chapter 4: Black–Scholes model
Outline

Appendix

Proving the Black-Scholes-Merton Result

- Suppose that a tree with n time steps is used to value a European call option with strike price K and life T.
  - Each step is of length $\frac{T}{n}$.
  - If there have been j upward movements and $n-j$ downward movements on the tree, the final stock price is
  $$S_0u^jd^{n-j}$$
  where u is the proportional up movement, d is the proportional down movement, and S_0 is the initial stock price.
  - The payoff from a European call option is then
  $$\max(S_0u^jd^{n-j} - K, 0).$$
- From the properties of the binomial distribution, the probability of exactly j upward and $n-j$ downward movements is given by
$$\frac{n!}{(n-j)!j!}p^j(1-p)^{n-j}$$

Proving the Black-Scholes-Merton Result

- It follows that the expected payoff from the call option is

$$\sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$$

- As the tree represents movements in a risk-neutral world, we can discount this at the risk-free rate r to obtain the option price:

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0) \quad (1)$$

Proving the Black-Scholes-Merton Result

- The option is in the money when the final stock price is greater than the strike price, that is, when

$$S_0 u^j d^{n-j} > K$$

or

$$\ln\left(\frac{S_0}{K}\right) > -j \ln(u) - (n-j) \ln(d)$$

Proving the Black-Scholes-Merton Result

- Since $u = e^{\sigma\sqrt{\frac{T}{n}}}$ and $d = e^{-\sigma\sqrt{\frac{T}{n}}}$, this condition becomes

$$\ln\left(\frac{S_0}{K}\right) > -j\sigma\sqrt{\frac{T}{n}} - (n-j)(-\sigma\sqrt{\frac{T}{n}})$$

that is

$$\ln\left(\frac{S_0}{K}\right) > n\sigma\sqrt{\frac{T}{n}} - 2j\sigma\sqrt{\frac{T}{n}}$$

or

$$j > \frac{n}{2} - \frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{\frac{T}{n}}}.$$

Proving the Black-Scholes-Merton Result

- Equation (1) can therefore be written

$$c = e^{-rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} (S_0 u^j d^{n-j} - K)$$

where

$$\alpha \equiv \frac{n}{2} - \frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{\frac{T}{n}}}$$

Proving the Black-Scholes-Merton Result

- For convenience, we define

$$U_1 \equiv \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

and

$$U_2 \equiv \sum_{j>\alpha} \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j}$$

so that

$$c = e^{-rT} \left(S_0 U_1 - K U_2 \right). \quad (2)$$

- Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution.
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution.

Proving the Black-Scholes-Merton Result
Consider U_{2}

- As is well known, the binomial distribution approaches a normal distribution as the number of trials approaches infinity.
  - Specifically, when there are n trials and p is the probability of success, the probability distribution of the number of successes is approximately normal with mean np and standard deviation $\sqrt{np(1-p)}$.
- U_2 is the probability of the number of successes being more than α.
- From the properties of the normal distribution, it follows that, for large n,

$$U_2 = N\left(\frac{np - \alpha}{\sqrt{np(1-p)}} \right)$$

where N is the cumulative normal distribution function.

Proving the Black-Scholes-Merton Result
Consider U_{2}

- Substituting for α, we obtain

$$U_2 = N\left(\frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{T}\sqrt{p(1-p)}} + \frac{\sqrt{n}\left(p - \frac{1}{2}\right)}{\sqrt{p(1-p)}} \right)$$

Proving the Black-Scholes-Merton Result
Consider U_{2}

- From (see Chapter 11)

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

and

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

with $\Delta t = \frac{T}{n}$ we have

$$p = \frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}}$$

Proving the Black-Scholes-Merton Result
Consider U_{2}

- By expanding the exponential functions in a series, we see that, as n tends to infinity, $p(1-p)$ tends to $\frac{1}{4}$ and $\sqrt{n(p-\frac{1}{2})}$ tends to

$$\frac{(r-\frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$$

so that in the limit, as n tends to infinity, we finally obtain

$$U_2 = N\left(\frac{\ln\left(\frac{S_0}{K}\right)+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = N(d_2)$$

Proving the Black-Scholes-Merton Result
Consider U_{1}

- U_1 rewrites as

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (pu)^j \left[(1-p)d\right]^{n-j}$$

- Define

$$p^* \equiv \frac{pu}{pu+(1-p)d} \qquad (3)$$

Proving the Black-Scholes-Merton Result
Consider U_{1}

- It then follows that

$$1-p^* = \frac{(1-p)d}{pu+(1-p)d}$$

and

$$\begin{aligned} U_1 &= \sum_{j>\alpha} \frac{n!}{(n-j)!j!} \left(p^*\left(pu+(1-p)d\right)\right)^j \\ &\quad \left[(1-p^*)\left(pu+(1-p)d\right)\right]^{n-j} \\ &= \left[pu+(1-p)d\right]^n \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (p^*)^j (1-p^*)^{n-j} \end{aligned}$$

Proving the Black-Scholes-Merton Result
Consider U_{1}

- Since the expected return in the risk-neutral world is the risk-free rate r, it follows that

$$\left[pu+(1-p)d\right]^n = e^{rT}$$

and

$$U_1 = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!j!} (p^*)^j (1-p^*)^{n-j}$$

- This shows that U_1 involves a binomial distribution where the probability of an up movement is p^* rather than p.

Proving the Black-Scholes-Merton Result
Consider U_{1}

- Approximating the binomial distribution with a normal distribution, we obtain

$$U_1 = e^{rT} N\left(\frac{np^* - \alpha}{\sqrt{np^*(1-p^*)}}\right)$$

and substituting for α gives for U_2

$$U_2 = e^{rT} N\left(\frac{\ln\left(\frac{S_0}{K}\right)}{2\sigma\sqrt{T}\sqrt{p^*(1-p^*)}} + \frac{\sqrt{n}\left(p^* - \frac{1}{2}\right)}{\sqrt{p^*(1-p^*)}}\right)$$

Proving the Black-Scholes-Merton Result
Consider U_{1}

- Substituting for u and d in equation in equation (3) gives

$$p^* = \left(\frac{e^{r\frac{T}{n}} - e^{-\sigma\sqrt{\frac{T}{n}}}}{e^{\sigma\sqrt{\frac{T}{n}}} - e^{-\sigma\sqrt{\frac{T}{n}}}}\right)\left(\frac{e^{\sigma\sqrt{\frac{T}{n}}}}{e^{r\frac{T}{n}}}\right)$$

- By expanding the exponential functions in a series we see that, as n tends to infinity, $p^*(1-p^*)$ tends to $\frac{1}{4}$ and $\sqrt{n(p-\frac{1}{2})}$ tends to

$$\frac{(r + \frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$$

with the result that

$$U_1 = e^{rT} N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = e^{rT} N(d_1)$$

Proving the Black-Scholes-Merton Result

- Finally, from equation (2) we have

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

that is

$$c = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2).$$

QED.