Industrial Organization

Master Quantitative Economics - 2023/2024

Chapter 1: Static Models of Oligopoly

Jérôme MATHIS

www.jeromemathis.fr/MQE

password: dauphine-MQE

Univ. Paris-Dauphine

Chapter 1

Static Models of Oligopoly
Outline

Introduction
Bibliography

English ed.: MIT; French ed.: Economica

Introduction
Jean Tirole (Nobel 2014)

Affiliation: Toulouse School of Economics (TSE), Toulouse, France

- **Prize motivation:** "for his analysis of market power and regulation"
- **Field:** industrial organization, microeconomics

Introduction
Issue

- Questions:
  - What is the price on a given market?
  - What are the profits?
  - What is the social surplus?
- Answers. It depends on:
  - How many firms are on the market
    - Monopoly, duopoly, oligopoly, ... , atomless firms.
  - Whether firms are competing on prices or on quantity.
  - Whether there are capacity constraints, decreasing returns to scale, ...
  - Whether there is a temporal dimension, product differenciation, ...

Introduction
Issue

- You already know that:
  - profit is maximal under monopoly
    - price is chosen such that profit is maximal
  - profit is minimal under pure and perfect competition
    - price equals marginal cost.

Static Models of Oligopoly
Outline

Bertrand Paradox
Introduction

Joseph Louis François Bertrand (1822-1900)

Bertrand Paradox
Model

- $N = \{1, 2\}$: Two firms produce goods that are perfect substitutes in the consumers' utility functions.
- The market demand function is

$$q = D(p)$$

and the demand for the output of firm i, $i \in N$, denoted as D_i, is

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & \text{if} \quad p_i < p_j \\ \frac{D(p_i)}{2} & \text{if} \quad p_i = p_j \\ 0 & \text{otherwise} \end{cases}$$

- Each firm incurs a cost c per unit of production.
- So the profit of firm i is:

$$\pi_i(p_i, p_j) = (p_i - c) D_i(p_i, p_j)$$

Bertrand Paradox
Results

Proposition (Bertrand (1883))

The unique equilibrium has the two firms price at marginal cost and do not make profits.

Proof.

Assume (p_1^*, p_2^*) is an equilibrium. Let us show that $p_1^* = p_2^* = c$ Assume $p_k^* = c$. By charging $p_j^* \neq c$, firm $j \neq k$ makes either zero profits (if $p_j^* > c = p_k^*$) or negative profits (if $p_j^* < c = p_k^*$). By charging $p_j^* = c$ firm j makes zero profits and there is no profitable deviation. □

Question

Is the proof finished?

Bertrand Paradox
Results

Answer

No! We still have to show that this equilibrium is unique.

Proof.

Per contra, we shall show in all following cases that the firm k ($k \in \{1, 2\}$ to be specified) would increase its profits by charging a price $p_k \neq p_k^*$.

First case: $\min\{p_1^*, p_2^*\} < c$. Takes $k = \arg\min_{i \in N}\{p_i^*\}$.

So firm k makes strictly negative profits.

A profitable deviation is to charge a higher price $p_k = c > p_k^*$. □

Bertrand Paradox
Results

Proof.

Second case: $\min\{p_1^*, p_2^*\} > c$. Takes $k = \arg\max_{i \in N}\{p_i^*\}$.

A profitable deviation for firm k is to charge a lower price that is slightly below the competitor's one $p_k = p_j - \varepsilon$, $j \neq k$, $\varepsilon > 0$.

For ε small enough, the new price p_k is still higher than c so the resulting profit is strictly positive. □

Bertrand Paradox
Results

Proof.

Third case: $\min\{p_1^*, p_2^*\} = c$. Then $\max\{p_1^*, p_2^*\} > c$. Takes $k = \arg\min_{i \in N}\{p_i^*\}$.

Firm k has a profitable deviation to charge a higher price that is slightly below the competitor's one $p_k = p_j - \varepsilon, j \neq k, \varepsilon > 0$ and small enough. □

Bertrand Paradox
Results

Question

What happens in the asymmetric case where firm 1 has lower marginal cost $c_1 < c_2$?

Proposition

When $c_1 < c_2$:

- firm 2 makes no profit; and

- firm 1 charges price $p = c_2$ and makes a profit of $(c_2 - c_1) D(c_2)$

(as long as $c_2 \leq p^m(c_1) \in \arg\max_p (p - c_1) D(p)$; otherwise firm 1 charges its monopoly price $p^m(c_1)$).

Bertrand Paradox
Results

Intuition

Firm 1 charges an ε below c_2 to make sure it has the whole market.

Remark (1)

In fact, there are equilibria where firm 1 charges c_2 (not an ε-below).

These rely on firm 2 randomizing uniformly over $[c_2, c_2 + \eta]$, for small enough $\eta > 0$. See, Blume (2003).

Remark (2)

Beyond this existence result, we "almost" have uniqueness:

In every Nash equilibrium in which firms use undominated strategies, the low-cost firm 1 serves the entire market at a price equal c_2.

See Kartik (2011).

Bertrand Paradox
Results

- If there are n firms, each with a constant marginal cost satisfying $c_1 = c_2 = ... = c_{n-1} < c_n$ then $p^* = c_1$ and consumers are distributed among firm 1 to $n - 1$.

Bertrand Paradox
Conclusion

- When competing in prices, two firms (having the same marginal costs) is enough to replicate the pure and perfect competition.
- We have seen:
  - $c_1 = c_2 \Longrightarrow p^* = c_1$ and $\pi_1 = \pi_2 = 0$ (p.p.c.)
  - $c_1 < c_2 \Longrightarrow p^* = c_2$ and $\pi_1 > 0 = \pi_2$ (non p.p.c.)
  - $c_1 = c_2 < c_3 \Longrightarrow p^* = c_1$ and $\pi_1 = \pi_2 = \pi_3 = 0$ (p.p.c.).

Bertrand Paradox
Extension

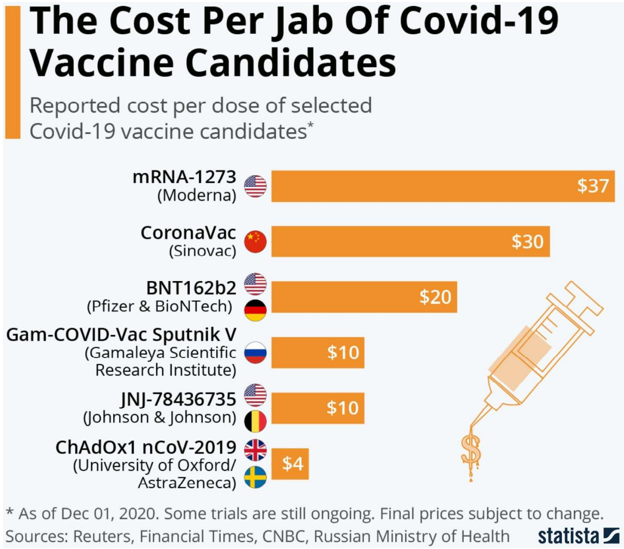
Question

Is the Bertrand Paradox robust when introducing capacity constraints?

Answer

No!

Bertrand Paradox
Extension



Bertrand Paradox
Extension

- Assume that firm 1 has a production capacity smaller than $D(c)$.

Question

Is $(p_1^*, p_2^*) = (c, c)$ still an equilibrium price system?

Answer

No, because if firm 2 increases its price slightly, it has a residual non-zero demand (since firm 1 cannot satisfy $D(c)$). So, firm 2 makes positive profits.

Bertrand Paradox
Extension

- The form of the residual-demand depends on which consumers are served by the low-price firm 1.
- Let us consider some decreasing returns to scale.
  - $C_i(q_i)$ is increasing and convex: $C_i' > 0$ and $C_i'' < 0$.
  - This is a generalization of capacity constraints (see Figure 4.4)

Bertrand Paradox
Extension

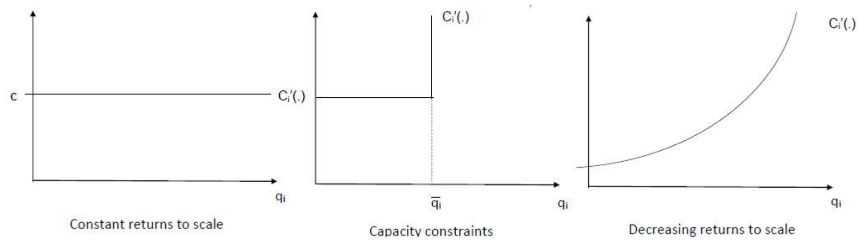


Figure 4.4

Bertrand Paradox
Extension

- At a given price p, a firm is not willing to supply more than its competitive supply $S_i(p) \in \arg\max_q \pi(p,q) = \arg\max\{pq - C_i(q)\}$ which is defined by
$$p = C_i'(S_i(p))$$
- Assume that firm 1 has a capacity constraint, i.e., $S_1(p) < D(p)$ and $p_1 < p_2$.
  - So firm 2 faces some residual demand.

Question

If we want to maximize the consumers surplus which consumers shall we serve?

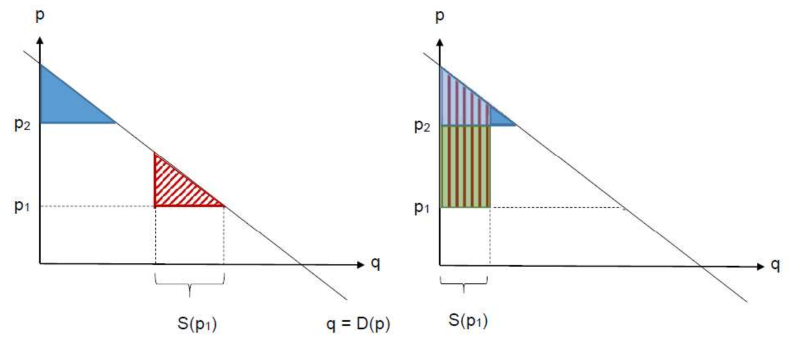
Bertrand Paradox
Extension

Answer

The most eager consumers!

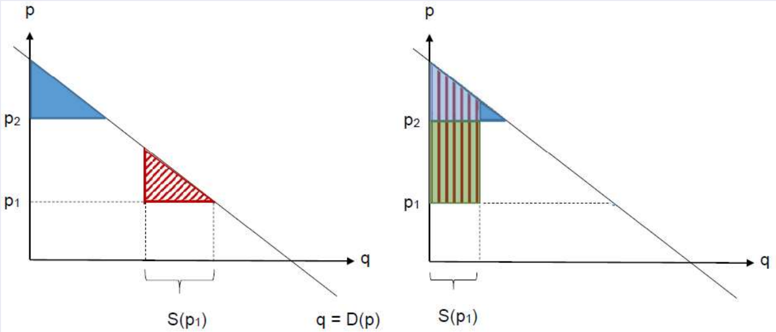
Proof.

[(Sketch)] □



Bertrand Paradox
Extension

Proof.

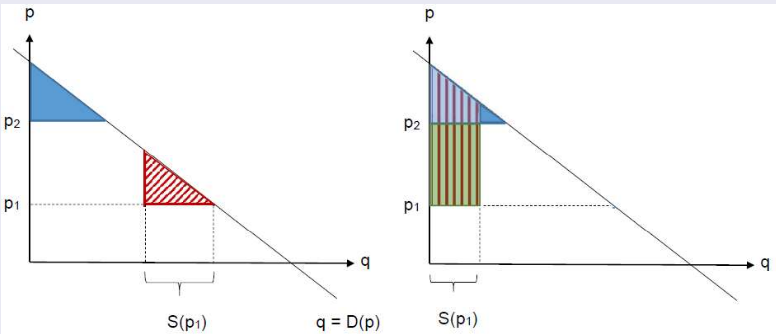


On the LHS (resp. RHS) of Figure 4.5, the two areas (red and blue) depicts the total consumer surplus when serving the least (resp. most) eager agents (...) □

Bertrand Paradox
Extension

Proof.

[(Sketch)]

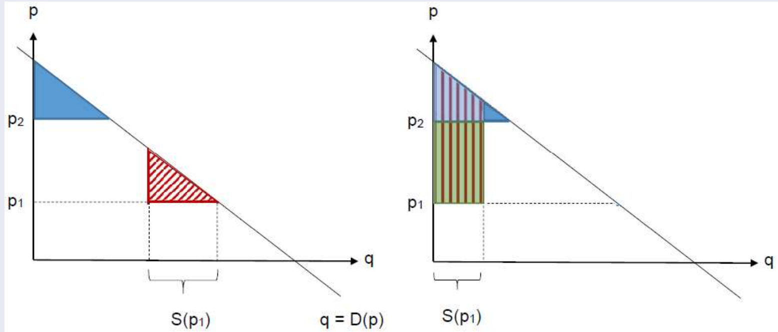


In all cases, we have the red area because in all cases consumers with a valuation for the good higher than price p_2 will be served (...) □

Bertrand Paradox
Extension

Proof.

[(Sketch)]



In the second case, we have in addition the green area defined by the rectangular $(p_2 - p_1)S(p_1)$. □

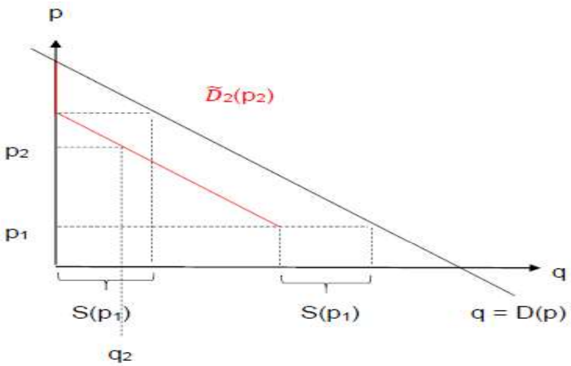
Bertrand Paradox
Extension

- This rationing is then called the *efficient-rationing* rule.
  - It seems quite strange because if $D(p_2) < S(p_1)$, when serving the most eager consumers firm 2 will not sell anything.
  - While when serving the least eager, firm 1 will sell $S(p_1)$ and firm 2 will sell $D(p_2)$.
  - But, as we see with the green area the surplus is higher when serving the most eager consumers.
  - Note that it would be obtained if the consumers were able to costlessly resell the good to each other.

Bertrand Paradox
Extension

- The efficient-rationing rule defines a residual function for firm 2:

$$\tilde{D}_2(p_2) = \begin{cases} D(p_2) - S(p_1) & \text{if} \quad D(p_2) > S(p_1) \\ 0 & \text{otherwise} \end{cases}.$$



Bertrand Paradox
Extension

- The *Proportional* or *Randomized-rationing* rule provide all consumers with the same probability of being rationed.
  - The probability of not being able to buy from firm 1 is:
  - $$\frac{D(p_1) - S(p_1)}{D(p_1)}$$
  - Hence, the residual demand facing firm 2 is:

$$\tilde{D}_2(p_2) = D(p_2)\left(\frac{D(p_1) - S(p_1)}{D(p_1)}\right).$$

Bertrand Paradox
Extension

Question

How to draw it?

Answer

$\tilde{D}_2(\cdot)$ is linear since $D(p_2)$ is linear in p_2 and for a fixed p_1, $\left(\frac{D(p_1)-S(p_1)}{D(p_1)}\right)$ is a constant.
Then we only need to know two points.
1) $D(p_2) = 0 \Longrightarrow \tilde{D}_2(p_2) = 0$
2) $\tilde{D}_2(p_1) = D(p_1) - S(p_1)$.

Bertrand Paradox
Extension

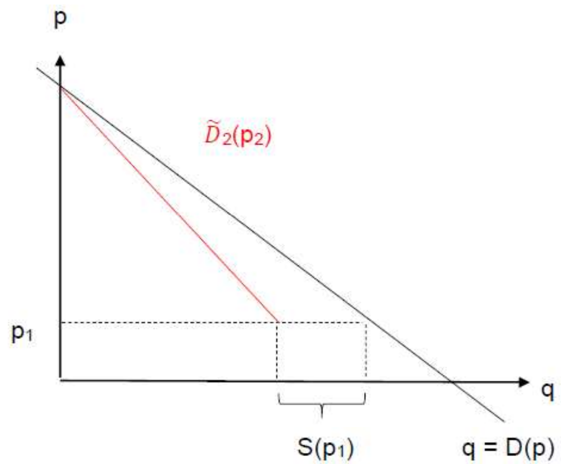


Figure 4.7

Bertrand Paradox
Extension

Question

Which rule firm 2 prefers? Said differently, under which rule firm 2's residual demand is higher at each price?

Answer

The second rule!

- We can see that graphically by comparing Figures 4.6 and 4.7.
  - And also analytically:

$$
\begin{aligned}
p_1 &< p_2 \Rightarrow D(p_1) > D(p_2) \Rightarrow -D(p_1)\,S(p_1) < -D(p_2)\,S(p_1) \\
&\Rightarrow D(p_1)\,D(p_2) - D(p_1)\,S(p_1) < D(p_1)\,D(p_2) - D(p_2)\,S(p_1) \\
&\Rightarrow D(p_2)\left(\frac{D(p_1) - S(p_1)}{D(p_1)}\right) > D(p_2) - S(p_1).
\end{aligned}
$$

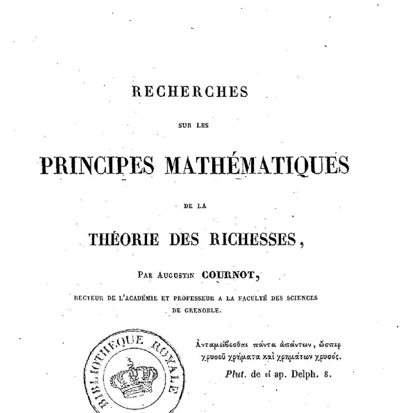
Static Models of Oligopoly
Outline

Cournot Market Structure
Introduction

Antoine Augustin Cournot (1801-1877)

Cournot Market Structure
Introduction

- Cournot first outlined his theory of competition in 1838.
  - *Recherches sur les Principes Mathematiques de la Theorie des Richesses*
  - Contains explicit and mathematically precise models.


RECHERCHES
SUR LES
PRINCIPES MATHÉMATIQUES
DE LA
THÉORIE DES RICHESSES,
PAR AUGUSTIN COURNOT,
RECTEUR DE L'ACADÉMIE ET PROFESSEUR A LA FACULTÉ DES SCIENCES DE GRENOBLE.

Cournot Market Structure
Introduction

- Cournot described the competition with a market for spring water dominated by two suppliers (a duopoly).
  - He constructed profit functions for each firm
  - He then used partial differentiation to construct a function representing a firm's best response for given output levels of the other firm(s) in the market.
  - He showed that a stable equilibrium occurs where these functions intersect (i.e. the simultaneous solution of the best response functions of each firm).
  - In equilibrium, each firm's expectations of how other firms will act are shown to be correct; when all is revealed, no firm wants to change its output decision.
- This idea of stability was later taken up and built upon as a description of Nash equilibria, of which Cournot equilibria are a subset.
  - Cournot equilibrium (1838) is a Nash equilibrium (1950).

Cournot Market Structure
General setting

- $N = \{1, 2, ..., n\}$: n firms who:
  - produce a homogeneous product;
  - do not cooperate, i.e. there is no collusion;
  - have market power, i.e. each firm's output decision affects the good's price;
  - compete in quantities, and choose quantities simultaneously;
    - ⋆ E.g., oil extraction (if OPEC was not a cartel), agricultural products (sugar, cocoa, ...)
  - are rational and act strategically
    - ⋆ They seek to maximize profit given their competitors' decisions.

Cournot Market Structure
General setting

- Each firm $i \in N$:
  - has a production cost $C_i(q_i)$.
  - uses its production level $q_i \in \mathbb{R}$ as a strategy.
  - takes the quantity set by its competitors as a given, evaluates its residual demand, and then behaves as a monopoly.

Cournot Market Structure
General setting

- Total output $Q = \sum_{i=1}^{n} q_i$.
  - We denote $Q_{-i} := Q - q_i = \sum_{j=1, j \neq i}^{n} q_j$.
- Price adjusts to clear the market: $p = P(Q)$.
- Firm i's profit:

$$\pi_i(q_i, Q_{-i}) = q_i P(q_i + Q_{-i}) - C_i(q_i)$$

Cournot Market Structure
General setting

Definition

A profile $(q_1^*, q_2^*, ..., q_n^*)$ is a **Cournot equilibrium** if for all $i \in N$, we have

$$q_i^* \in \arg\max_{q_i} \pi_i\left(q_i, Q_{-i}^*\right)$$

with $Q_{-i}^* := \sum_{j=1, j\neq i}^{n} q_j^*$.

- At Cournot equilibrium, each firm maximizes its profit given the quantity chosen by the other firms.
  - So, Cournot equilibrium (1838) is a Nash equilibrium (1950).
  - It is also called a **(pure-strategy) Cournot-Nash equilibrium**.

Cournot Market Structure
General setting

- F.O.C.

$$\frac{\partial \pi_i\left(q_i, Q_{-i}\right)}{\partial q_i} = 0 \Longleftrightarrow \frac{\partial}{\partial q_i}\left(q_i P\left(q_i + Q_{-i}\right) - C_i\left(q_i\right)\right) = 0$$
$$\Longleftrightarrow \left[P\left(q_i + Q_{-i}\right) - C_i'\left(q_i\right)\right] + \left[q_i P'\left(q_i + Q_{-i}\right)\right] = 0$$

  - The first bracket denotes the profitability of an extra unit of output
    - I.e., difference between price and marginal cost.
  - The second bracket denotes the profitability of inframarginal units
    - I.e., extra unit creates a decrease in price P', which affects the q_i units already produced.

Cournot Market Structure
General setting

- For a competitive firm $P'(\cdot) = 0$ because the firm is too small to affect the market price.
  - So, FOC writes as
    $$P\left(q_i + Q_{-i}\right) = C_i'\left(q_i\right)$$
  - The firm prices at marginal cost.
- For a monopoly, $q_i = Q$ and $Q_{-i} = 0$
  - So, FOC writes as
    $$P(Q) + P'(Q)\, Q = C_i'(Q)$$
  - The monopoly chooses a price such that the marginal revenue (LHS) equals the marginal cost (RHS).

Cournot Market Structure
General setting

- F.O.C.
  $$P\left(q_i + Q_{-i}\right) - C_i'\left(q_i\right) + q_i P'\left(q_i + Q_{-i}\right) = 0$$
- The FOC illustrates the negative externality between the firms:
  - when choosing its output, firm i takes into account the adverse effect of the market price on its own output
    - I.e., by considering $q_i P'(Q)$
  - rather than the effect on aggregate output
    - I.e., by considering $QP'(Q)$.

Cournot Market Structure
General setting

- Hence each firm will tend to choose an output that exceeds the optimal output from the industry point of view (since $Q_{-i}P'(Q) < 0$).
- Thus the market price will be lower than the monopoly price.
- Also, the aggregate profit will be lower than the monopoly profit.

Cournot Market Structure
General setting

- FOC can be rewritten as the Lerner index (1934) which describes the firm i's market power:

$$L_i := \frac{P - C_i'(q_i)}{P}$$

with $L_i \in [0, 1]$ (higher index implies greater market power; $L_i = 0$ means no market power at all).

Cournot Market Structure
General setting

- By introducing the price-elasticity of demand facing firm i:

$$\varepsilon(p) := \frac{dD}{dp}\frac{p}{D} = p\frac{D'(p)}{D(p)}$$

which has the interpretation that p increasing by 1% yields the quantity demanded increases by ε%.

  - Note that economists often refer to price-elasticity of demand as a positive value (i.e., in absolute value terms: $\varepsilon(p) := -p\frac{D'(p)}{D(p)}$) with the interpretation that p increasing by 1% yields the quantity demanded *decreases* by ε%.

Cournot Market Structure
General setting

- It is sometimes useful to rewrite Lerner index as a function of the individual market share $\frac{q_i}{Q}$ and elasticity:

$$L_i = \frac{P - C_i'(q_i)}{P} = -\frac{q_i}{Q}\frac{1}{\varepsilon(P)}$$

The second equality comes from our previous F.O.C. according to which $P - C_i'(q_i) + q_i P' = 0$, so
$\frac{P - C_i'(q_i)}{P} = -\frac{q_i P'}{P} = -\frac{q_i\left(\frac{dP}{dD}\right)}{P} = \frac{q_i\left(\frac{P}{-\varepsilon(P)D}\right)}{P} = q_i\frac{1}{-\varepsilon(P)Q}.$

Cournot Market Structure
General setting

- $L_i > 0$ since $D'(p) < 0 \Longrightarrow \varepsilon(p) < 0$.
  - So firms sells at a price exceeding marginal cost.
  - Thus, the Cournot equilibrium is not socially efficient.

Cournot Market Structure
Linear Model: n firms with possibly non identical marginal costs

- $D(p) = 1 - p$
- Constant return to scale: $C_i(q_i) = c_i q_i$
- Each firm chooses q_i that solves

$$\max_{q_i} \left(\pi_i(q_i, Q_{-i})\right)$$

with

$$\pi_i(q_i, Q_{-i}) = (1 - q_i - Q_{-i})\, q_i - c_i q_i$$

Cournot Market Structure
Linear Model: n firms with possibly non identical marginal costs

- Assuming $q_i > 0$ for all $i \in N$, FOC is

$$\begin{aligned} & 1 - 2q_i - Q_{-i} = c_i \\ \Longleftrightarrow\quad & 1 - q_i - Q = c_i \end{aligned}$$

- Summing over all q_i yields:

$$n - Q - nQ = \sum_{i=1}^{n} c_i$$

Cournot Market Structure
Linear Model: n firms with possibly non identical marginal costs

- Thus the Cournot equilibrium aggregate industry output and market price are

$$Q = \tfrac{n - \sum_{i=1}^{n} c_i}{n+1} \quad \text{and} \quad p = 1 - Q = \tfrac{1 + \sum_{i=1}^{n} c_i}{n+1}$$

- Also, we find

$$\begin{aligned} q_i &= 1 - Q - c_i = p - c_i = \frac{1 + \sum_{i=1}^{n} c_i}{n+1} - c_i \\ &= \frac{1 + \sum_{j \neq i} c_j - nc_i}{n+1} \end{aligned}$$

- So a firm's output decreases with its marginal cost and increases with its competitors' marginal costs.

Cournot Market Structure
Linear Model: Duopoly

- From the previous section, when $n = 2$, we get:
  - the firm j's reaction curve write as:

$$q_j(q_i) = \frac{1 - c_j - q_i}{2}$$

  - the Cournot equilibrium firm j's output writes as:

$$\begin{aligned} q_j^* &= q_j^*\left(q_i^*\left(q_j\right)\right) = \frac{1 - c_j}{2} - \left(\frac{1 - c_i - q_j}{2}\right) \\ &= \frac{1 + c_i - 2c_j}{3} \end{aligned}$$

Cournot Market Structure
Linear Model: Duopoly

- We can depict the reaction curves in the (q_1, q_2) space:

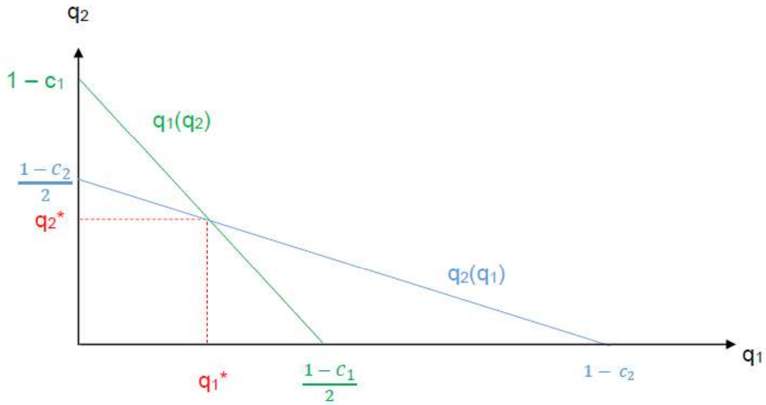


Figure 4.10

Cournot Market Structure
Linear Model: Duopoly

Question

What would be the effect of an increase in firm 1's marginal cost?

Answer

It would have the effect of decrease firm 1's output and increase firm 2's output.

- Indeed,...

Cournot Market Structure
Linear Model: Duopoly

- Indeed, if $c_1 \to c_1' > c_1$ we get $q_1'^* < q_1^*$ and $q_2'^* > q_2^*$.

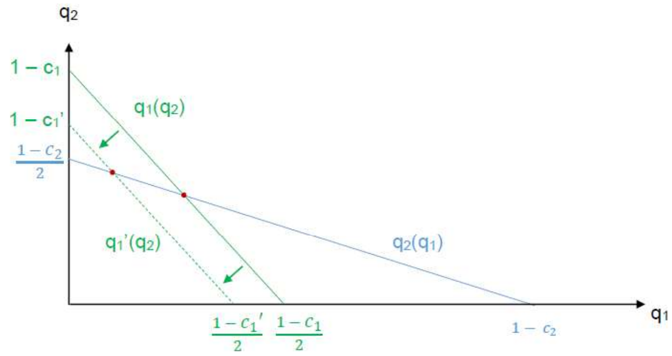


Figure 4.11

Cournot Market Structure
Linear Model: n firms with identical marginal costs

- $c_i = c$ for all $i \in N$
- We then obtain a symmetric equilibrium (i.e., $q_i^* = q$, for all $i \in N$) given by:

$$\frac{p-c}{p} = \frac{1}{n}\frac{1}{\varepsilon(p)} \quad \text{and} \quad Q = nq = n\frac{1-c}{n+1}$$

and

$$q = \frac{1-c}{n+1} \quad ; \quad p = 1 - nq = c + \frac{1-c}{n+1} \quad \text{and} \quad \pi_i = \pi = \frac{(1-c)^2}{(n+1)^2}$$

- Varying the number of firms:
  - $n = 1$: monopoly situation;
  - $n \longrightarrow +\infty$: $\lim_{n \longrightarrow +\infty} Q = 1 - c$ and $\lim_{n \longrightarrow +\infty} p = c$, competitive solution.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- F. H. Hahn (1962): "The Stability of the Cournot Oligopoly Solution", *The Review of Economic Studies*, Vol. 29, No. 4, pp. 329-331

Definition

Firm i's **reaction function** is defined by $R_i : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ with

$$R_i(Q_{-i}) := \arg\max_{q_i} \pi_i(q_i, Q_{-i}).$$

- So,

$$R_i(Q_{-i}) = \arg\max_{q_i} \{q_i P(q_i + Q_{-i}) - C_i(q_i)\}.$$

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- Observe that the assumption $R_i : \mathbb{R} \longmapsto \mathbb{R}$, means that firm i only focuses on the total quantity $Q_{-i} \in \mathbb{R}$.
  - Firm i could rather take into account on which competitor produces what quantity.
    - ⋆ We then would have $R_i : \mathbb{R}^{n-1} \longmapsto \mathbb{R}$with $R_i\left((q_j)_{j \neq i}\right)$.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- We can now rewrite the definition of Cournot equilibrium wrt reaction functions.

Definition

A profile $(q_1, q_2, ..., q_n)$ is a (pure-strategy) **Cournot-Nash equilibrium** if for all $i \in N$, we have

$$q_i = R_i(Q - q_i)$$

with $Q := \sum_{i=1}^n q_i$.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- Said differently, such an equilibrium is obtained by finding an aggregate output such that

$$Q = \sum_{i=1}^{n} q_i(Q)$$

that is, a fixed point of the function

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

where $q_i(Q)$ solves $P(Q) + q_i P'(Q) - C_i(q_i) = 0$ or is equal to zero if this equation has no positive solution.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

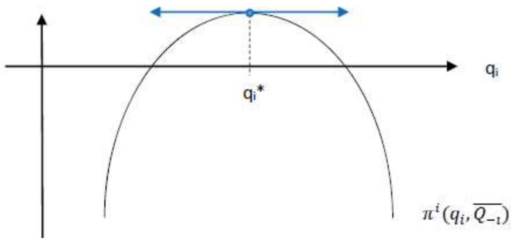
- If $\pi_i(q_i, Q_{-i})$ is strictly concave then the reaction function $R_i(\cdot)$ is:
  - (I.e., if $\frac{\partial^2 \pi_i(q_i, Q_{-i})}{\partial q_i^2} = 2P'(q_i + Q_{-i}) + q_i P''(q_i + Q_{-i}) - C_i''(q_i) < 0$.)
  - continuous, single-valued and defined by the FOC.
  - 

Figure 4.12

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- The FOC writes as

$$\frac{\partial \pi_i(q_i, Q_{-i})}{\partial q_i} = 0$$

which rewrites as

$$P(q_i + Q_{-i}) + q_i P'(q_i + Q_{-i}) - C_i'(q_i) = 0$$

that is, since $R_i(Q_{-i}) := \arg\max_{q_i} \pi_i(q_i, Q_{-i})$,

$$P(R_i(Q_{-i}) + Q_{-i}) + R_i(Q_{-i}) P'(R_i(Q_{-i}) + Q_{-i}) - C_i'(R_i(Q_{-i})) = 0$$

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- $R_i(Q_{-i})$ is decreasing if

$$\frac{\partial}{\partial Q_{-i}} \left(\frac{\partial \pi_i(q_i, Q_{-i})}{\partial q_i} \right) < 0$$

that is

$$\frac{\partial^2 \pi_i(q_i, Q_{-i})}{\partial q_i \partial Q_{-i}} < 0$$

Definition

Hahn conditions are:

$$\frac{\partial^2 \pi_i (q_i, Q_{-i})}{\partial q_i \partial Q_{-i}} < 0$$

and

$$P'(q_i + Q_{-i}) - C_i''(q_i) < 0$$

Proposition

Under Hahn conditions the Cournot equilibrium exists and is unique.

Proof.

First Hahn condition write as

$$\frac{\partial^2 \pi_i (q_i, Q_{-i})}{\partial q_i \partial Q_{-i}} < 0$$

that is

$$P'(q_i + Q_{-i}) + q_i P''(q_i + Q_{-i}) < 0$$

Summing this, to the second Hahn condition

$$P'(q_i + Q_{-i}) - C_i''(q_i) < 0$$

we obtain

$$2P'(q_i + Q_{-i}) + q_i P''(q_i + Q_{-i}) - C_i''(q_i) < 0$$

Proof.

which means that

$$\frac{\partial^2 \pi_i (q_i, Q_{-i})}{\partial q_i^2} < 0$$

So $R_i(\cdot)$ is continuous, single-valued and decreasing.
So is

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

The Brouwer theorem asserts that a continuous function from a compact set into itself admits at least one fixed point. □

Proof.

Here, compactness is easily obtained from

$$+\infty > q_i(0) \geq q_i(Q) \geq 0$$

where:
- the first inequality comes from the fact that each firm would produce a finite quantity if it were a monopoly;
- the second inequality comes from the fact that $q_i(\cdot)$ is decreasing; and
- the third one from the definition of q_i.

The equilibrium then exists. □

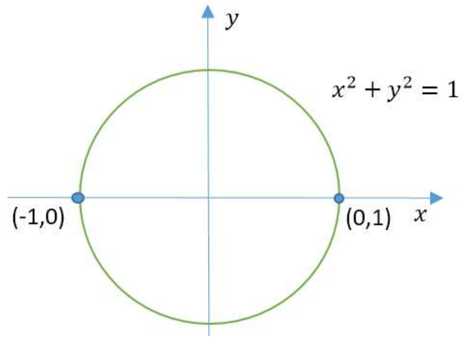
Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- To establish uniqueness, we need to apply the *Implicit Function Theorem.*

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- A function assigns a *single* value in the range for every value in the domain.
  - It is really covenient because we generally know how to compute the derivative and the integral of it.
- The problem is that some mathematical objects are not a function.
  - E.g., a circle defined by
  $$x^2 + y^2 = 1$$
  is not a function even though it describe a relationship between x and y.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium



- The idea of the *Implicit Function Theorem* is to use the fact that almost every point can locally (i.e., in a neighborhood) be described as a function.
  - E.g., only two points in our circle cannot: $(-1, 0)$ and $(1, 0)$.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- The *Implicit Function Theorem* provides conditions under which a relationship (not necessarily a function) of the form $F(x, y) = 0$ can be rewritten as a function $y = f(x)$ locally (in a small neighborhood of a point).
  - E.g., our circle can be described by the relationship $F(x, y) = x^2 + y^2 - 1$, which in turns for positive y take the form of $y = \sqrt{1 - x^2}$, and for negatives y takes the form of $y = -\sqrt{1 - x^2}$.
  - The Theorem is called *implicit* because it does not provide us with the explicit formulae of the function $f(\cdot)$, but rather just ensures its existence.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

Theorem (*Implicit Function Theorem*)

Let $F(x,y) \in C^1$ in a neighborhood of (x_0, y_0) such that:

$$F(x_0, y_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y}(x_0, y_0) \neq 0$$

Then there exists a neighborhood of (x_0, y_0) in which there is an implicit function $y = f(x)$ such that:
(i). $f(x_0) = y_0$;
(ii). $F(x, f(x)) = 0$ for every x in the neighborhood; and
(iii). $f'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ in the neighborhood.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- The Theorem can be rewritten as:
  - Let

$$\begin{array}{rccc} F: & E \times F & \longmapsto & G \\ & (x,y) & \longmapsto & F(x,y) \end{array}$$

    there is a function $f: E' \longmapsto F'$, with $E' \subset E$ and $F' \subset F$ such that

$$\begin{array}{rccc} F: & E' \times F' & \longmapsto & G \\ & (x,y) & \longmapsto & F(x, f(x)) \end{array}$$

  - $F(x, f(x)) = 0$ implies that $0 = F'_x(\cdot,\cdot) + \frac{\partial F(x,f(x))}{\partial f(x)} f'(x)$ so $f'(x) = -\frac{F'_x(\cdot,\cdot)}{F'_y(\cdot,\cdot)}$.

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

Proof.

By considering the function

$$F(x, f(x)) = \frac{\partial \pi_i (Q_{-i}, R_i(Q_{-i}))}{\partial R_i}$$

we obtain

$$\begin{aligned} R'_i(Q_{-i}) &= -\frac{\frac{\partial^2 \pi_i (Q_{-i}, R_i(Q_{-i}))}{\partial Q_{-i} \partial R_i}}{\frac{\partial^2 \pi_i (Q_{-i}, R_i(Q_{-i}))}{\partial R_i^2}} \\ &= -\frac{1}{1 + \frac{P'(q_i + Q_{-i}) - C''_i(q_i)}{P'(q_i + Q_{-i}) + q_i P''(q_i + Q_{-i})}} \in (-1, 0) \end{aligned}$$

□

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

Proof.

Now from $q_i(Q) = R_i(Q - q_i(Q))$ we obtain

$$q'_i(Q) = R'_i(Q - q_i(Q)) \times (1 - q'_i(Q))$$

so

$$q'_i(Q) = \frac{R'_i(Q - q_i(Q))}{1 + R'_i(Q - q_i(Q))}$$

which is negative since $R'_i(Q - q_i(Q)) \in (-1, 0)$.
Thus,

$$\varphi : Q \longmapsto \sum_{i=1}^{n} q_i(Q)$$

is strictly decreasing in Q and the equilibrium is unique. □

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- Inexistence due to discountinuity

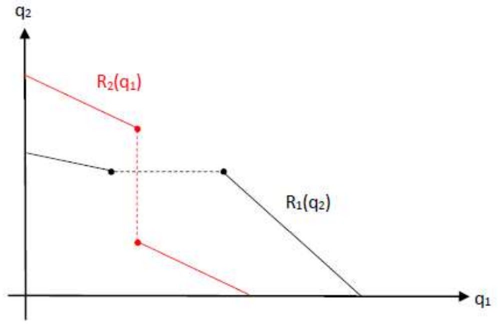


Figure 4.14

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- Multiplicity

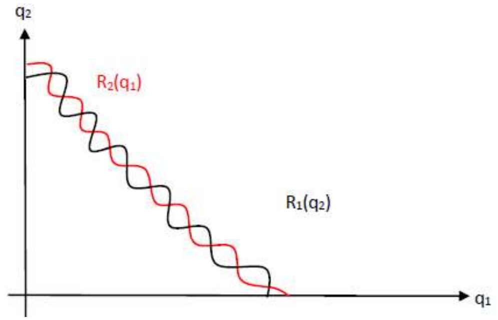


Figure 4.15

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- To be parallel R_i and R_j must be such that $R_i' = \frac{1}{R_j'}$ that is excluded by Hahn conditions (since $R_i' \in (-1, 0)$).

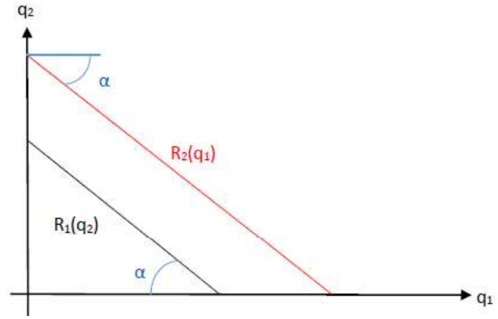


Figure 4.16
Here, $R_2' = \alpha$ and $R_1' = \frac{1}{\alpha}$

Cournot Market Structure
Existence and uniqueness of the Cournot equilibrium

- Regular Cournot equilibrium
  - $R_j^{-1}(0) > q_i^m$ $(= R_i(0))$: firm i's output that induces firm j to produce nothing exceeds firm i's monopoly output.

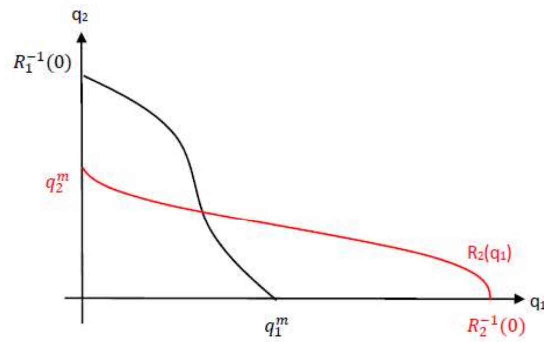


Figure 4.17

Cournot Market Structure
Conclusion

- Cournot outcome (output and price) is somewhere in between monopoly and perfect competition.
  - Aggregate output (resp. price) is greater (resp. lower) with Cournot duopoly than monopoly.
  - Aggregate output (resp. price) is lower (resp. greater) with Cournot duopoly than perfect competition.
- Firms have an incentive to form a cartel, effectively turning the Cournot model into a Monopoly.
  - Cartels are usually illegal, so firms might instead tacitly collude using self-imposing strategies to reduce output which, ceteris paribus will raise the price and thus increase profits for all firms involved.
  - We shall study it later...

Cournot Market Structure
Conclusion

- Cournot vs Bertrand:
  - Bertrand. More realistic assumption: firms compete in price (not quantity).
  - Cournot. More realistic prediction: two firms are not enough to push prices down to marginal cost level and then restore pure and perfect competition.
  - As the number of firms increases towards infinity, the Cournot model gives the same result as in Bertrand model
    - ⋆ The market price is pushed to marginal cost level.

Cournot Market Structure
Conclusion

- Neither model (Bertrand or Cournot) is necessarily better.
  - The accuracy of the predictions of each model will vary from industry to industry, depending on the closeness of each model to the industry situation.
  - If capacity and output can be easily changed, Bertrand is a better model of duopoly competition.
  - If output and capacity are difficult to adjust, then Cournot is generally a better model.
- We shall see later how to recast Cournot and Bertrand altogether as a two-stage model.

Static Models of Oligopoly
Outline

Stackelberg: sequential moves
Introduction

Heinrich Freiherr von Stackelberg (1905-1946)
Published *Market Structure and Equilibrium* (*Marktform und Gleichgewicht*) in 1934

Stackelberg: sequential moves
Model

- We consider a duopoly where firms move in sequence.
  - Firm 1: leader
  - Firm 2: follower
  - As in Cournot, competition is on quantity.
- Firms may engage in Stackelberg competition if one:
  - has some sort of advantage enabling it to move first;
  - is the incumbent monopoly of the industry and the follower is a new entrant;
  - is holding excess capacity; or
  - has commitment power.

Stackelberg: sequential moves
Result

Question
Is there any advantage for moving in the first stage rather than the second?

- We solve the game by backward induction.

Stackelberg: sequential moves
Result

- Second period subgames
  - Firm 2 chooses q_2 to maximize its profit given firm 1's quantity.
    - ⋆ Identical to the problem of firms in the Cournot market structure.
    - ⋆ Best response function of firm 2: $R_2(q_1)$.
- First period game

$$\max_{q_1} q_1 P\left(q_1 + R_2(q_1)\right) - C_1(q_1)$$

Stackelberg: sequential moves
Result

- FOC writes as

$$P(q_1 + R_2(q_1)) - C_1'(q_1) + q_1 P'(q_1 + R_2(q_1))(1 + R_2'(q_1)) = 0$$

- With respect to Cournot, we see that there is a new term in the FOC: $(1 + R_2'(q_1))$.
  - By adding this term, LHS becomes smaller than zero because $P'(\cdot) < 0$, and by Hahn conditions $R_2'(\cdot) \in (-1, 0)$.
  - So, we must decrease $R_2'(\cdot)$ to come back to zero.
  - Since $R_2'(\cdot) < 0$, we then must increase q_1.
  - Hence, $q_1^{Stackelberg} > q_1^{Cournot}$.

Stackelberg: sequential moves
Result

Question

Since the leader takes into account the follower's reaction function in his optimization program, why the leader does not behave as in the Cournot equilibrium?

Answer

Because moving sequentially is not as moving simultaneously.

Stackelberg: sequential moves
Result

Example

1\2	L	R
U	2,0	0,1
D	1,1	0,0

$BR^1(L) = \{U\}$; $BR^1(R) = \{U, D\}$; $BR^2(U) = \{R\}$; $BR^2(D) = \{L\}$.
So, the pure-strategy Nash equilibrium is (U, R).

When Player 1 moves at 1^{st}, the game becomes:

1\2	$BR^2(\cdot)$
U	0,1
D	1,1

and (D, L) is the Stackelberg outcome.

Stackelberg: sequential moves
Result

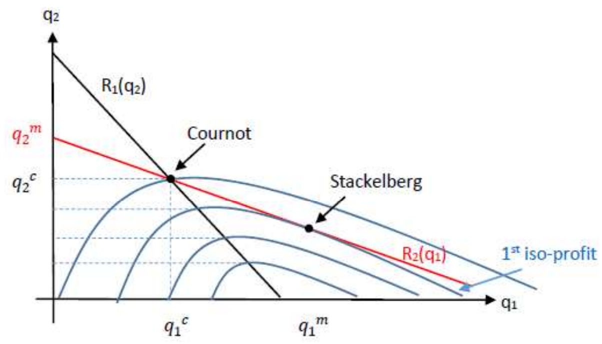


Figure 4.18

Stackelberg: sequential moves
Result

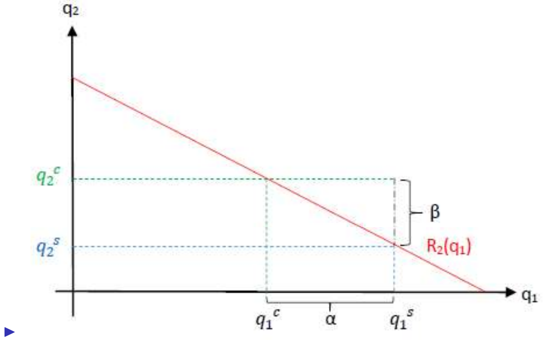
- Under Hahn conditions, $R_2'(\cdot) \in (-1, 0)$ so $q_1^{Cournot} + q_2^{Cournot} < q_1^{Stackelberg} + q_2^{Stackelberg}$.
  - 

Figure 4.19

Stackelberg: sequential moves
Conclusion

- Stackelberg outcome (output and price) is somewhere in between monopoly and perfect competition.
  - Aggregate output (resp. price) is greater (resp. lower) with Stackelberg than monopoly.
  - Aggregate output (resp. price) is lower (resp. greater) with Stackelberg than perfect competition.
- Stackelberg outcome is somewhere in between Cournot and Bertrand.
  - Aggregate output (resp. price) is greater (resp. lower) with Stackelberg than Cournot.
  - Aggregate output (resp. price) is lower (resp. greater) with Stackelberg than Bertrand.
  - Consumer surplus is greater (resp. lower) with Stackelberg than Cournot (resp. Bertrand).

Static Models of Oligopoly
Outline

Capacity and price game
Introduction

Question

How to recast Cournot and Bertrand altogether as a two-stage model?

David M. Kreps and Jose A. Scheinkman. “Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes”, *The Bell Journal of Economics*, Vol. 14, No. 2 (Autumn, 1983), pp. 326-337

Capacity and price game
Model

- $N = \{1, 2\}$
- 2 stages game:
  - Fimrs choose production capacities (i chooses $\bar{q}_i$)
  - Fimrs choose price (i chooses p_i)
- Demand function is concave
  - $P'(\cdot) < 0$ and $P''(\cdot) \leq 0$
- $C_i(q_i) = cq_i$ and $C_i(\bar{q}_i) = c_i\bar{q}_i$.
- Efficient rationing rule
  - $p_1 < p_2$ with $\bar{q}_1 < D(p_1)$ implies that the residual for firm 2 writes as

$$D(p_2, \bar{q}_1) = \max\{D(p_2) - \bar{q}_1, 0\}$$

Capacity and price game
Results

- To solve this two stages game, we proceed by backward induction.
  - We start by fixing the capacity constraint (1st period choices) to solve the resulting price game (2nd period choices).
  - Once the second stage best responses are characterized, we characterize the first stage best responses.

Capacity and price game
Results: The Price game

- Suppose firm i has a rigid capacity constraint $\bar{q}_i$

Proposition

Firms price at marginal cost if and only if $D(c) \leq \min\{\bar{q}_1, \bar{q}_2\}$.

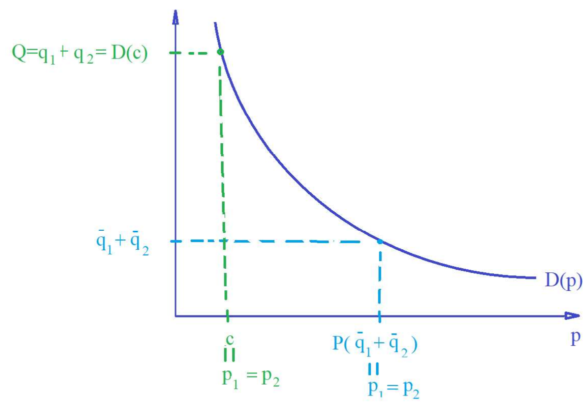
Proof.

"$\Longleftarrow$". This is Bertrand's Proposition
"$\Longrightarrow$". Assume, per contra, $p_1 = p_2 = c$ and $D(c) > \min\{\bar{q}_1, \bar{q}_2\} = \bar{q}_1$. Then firm 2 can set $p_2 = c + \varepsilon$ and face a positive residual demand, a contradiction. □

Capacity and price game
Results: The Price game

Proposition

In a pure-strategy equilibrium, firms sell up to capacity, i.e., $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$.



Capacity and price game
Results: The Price game

Proof.

Let us first show that $p_1 = p_2$.
Suppose, *per contra*, $p_1 < p_2$.
If $D(p_1) > \bar{q}_1$ then firm 1 woud be better-off by raising its price.
If $D(p_1) \leq \bar{q}_1$ then firm 1 supplies all the demand at a price $p_1 > c$ ($p_1 \leq c$ would allows firm 1 to increase its price above c and realize positive profit).
So we have $D(p_1) \leq \bar{q}_1$ and $p_1 > c$ which allows firm 2 to increase its profit by charging $p_2 = p_1 - \varepsilon$, with $\varepsilon \in (0, c - p_1)$, a contradiction. □

Capacity and price game
Results: The Price game

Proof.

Now let us show that $p = P(\bar{q}_1 + \bar{q}_2)$.
If $D(p) > \bar{q}_1 + \bar{q}_2$ then both firms ration their consumers. Each firm could increase its price and still sell its capacity.
If $D(p) < \bar{q}_1 + \bar{q}_2$ then the price is too high and one firm at least cannot sell its capacity. By charging $p - \varepsilon$ this firm would get all the market and sell its capacity. □

Capacity and price game
Results: The Price game

Proposition

There is a pure-strategy equilibrium in prices only if $\bar{q}_i \leq R_i(\bar{q}_j)$ for all i.

Proof.

Assume, per contra, $\bar{q}_i > R_i(\bar{q}_j)$ and a pure-strategy equilibrium in prices does exist.
By the previous Proposition, the pure-strategy equilibrium in prices satisfies $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$.
From $\bar{q}_i > R_i(\bar{q}_j)$ we have $P(\bar{q}_i + \bar{q}_j) < P(R_i(\bar{q}_j) + \bar{q}_j)$. So $p_i < P(\bar{q}_j + R_i(\bar{q}_j))$.
If firm i is capacity constrained then it can raises its price slightly and make profit $(p_i + \varepsilon)\bar{q}_i > p_i\bar{q}_i$. □

Capacity and price game
Results: The Price game

Proof.

If not, firm j must be capacity constrained (otherwise they would set lower prices). That is, $q_j = \bar{q}_j$. So by definition of the reaction function $R_i(\cdot)$, firm i's best responds by charging $p_i = P(R_i(\bar{q}_j) + \bar{q}_j)$, a contradiction. □

Capacity and price game
Results: The Price game

- Conclusion
  - According to the two first Propositions, for low capacities (*i.e.*, $\bar{q}_i \leq R_i(\bar{q}_j)$ for all i) we have the Cournot equilibrium outcome (*i.e.*, $p_1 = p_2 = P(\bar{q}_1 + \bar{q}_2)$).
  - According to the last Proposition, for high capacities (*i.e.*, $D(c) \leq \min\{\bar{q}_1, \bar{q}_2\}$) we have the Bertrand equilibrium outcome (*i.e.*, prices equals marginal cost).
  - For intermediate capacities we have no pure-strategy equilibrium. Further, the highest capacity firm makes a profit equal to its Stackelberg follower profit.
    - I.e., $\pi^F(\bar{q}_j) = R_i(\bar{q}_j)(P(R_i(\bar{q}_j) + \bar{q}_j) - c)$
    - See Kreps and Scheinkman (1983) for the complete proof, and Tirole (1988, MIT) for sketch).

Capacity and price game
Results: The Capacity game

- Let us now add a prior and simultaneous choice of capacities.
- Each firm has capacity $\bar{q}_i$ with cost $c_i\bar{q}_i$ and then decides to produce q_i with cost cq_i.
- Adding the capacity cost to the first period will not change the ssecond period reasonning because this cost is sunk.
- 2nd stage firm i's profit becomes:

$$\max_{q_i \leq \bar{q}_i} q_i \left(P\left(q_i + \bar{q}_j\right) - c\right) - c_i\bar{q}_i$$

- Given that firms sell up to capacity, the capacity choice $\bar{q}_i$ solves

$$\max_{\bar{q}_i} \bar{q}_i \left(P\left(\bar{q}_i + \bar{q}_j\right) - c - c_i\right)$$

Capacity and price game
Results: The Capacity game

Proposition

The Cournot outcome $(\bar{q}_1 = q^*, \bar{q}_2 = q^*)$ where q^* maximizes $q\left(P\left(q + q^*\right) - c - c_i\right)$ is an equilibrium.

Proof.

Suppose that firm i plays q^*. Firm j, if it plays $q \leq R(q^*)$ (where $R(\cdot)$ still denotes the 2^{nd} stage reaction function), by definition of q^*, gets

$$q\left(P\left(q + q^*\right) - c - c_i\right) \leq q^*\left(P\left(2q^*\right) - c - c_i\right)$$

If firm j plays $q > R(q^*)$ then it gets the Stackelberg follower profit.

$$\begin{aligned} \pi^F(q^*) &= R(q^*)\left(P\left(q^* + R(q^*)\right) - c - c_i\right) \\ &\leq q^*\left(P\left(2q^*\right) - c - c_i\right). \end{aligned}$$

□

Capacity and price game
Conclusion

- The Cournot equlibrium is the equilibrium in the 1^{st}-stage capacity game and the 2^{nd}-stage price is equal to $P(2q^*)$.
- The capacity game is a Cournot game with total producing costs $(c + c_i)\bar{q}_i$.
- To prove uniqueness in the choice of capacities require more work (see, Kreps and Scheinkman, 1983).

Capacity and price game
Conclusion

- Kreps and Scheinkman (1983) show that the difference between Cournot and Bertrand competition is more than just the strategy space, but that timing of decisions is also relevant.
  - To illustrate this, they study and solve a Bertrand like duopoly model of competition where timing of decision is inverted.
    - Capacity decision is made simultaneously and before price decision (as opposed to Bertrand models where the choice of capacity and price is interpreted as being simultaneous), and the low priced firm may not serve all the demand at her price (as it is in the Bertrand approach) due to capacity constraints.
    - In a two stage game where firms first set simultaneously capacity and then engage in simultaneous price competition with demand rationed following the efficient rationing rule, the unique Subgame Perfect Nash Equilibrium (SPNE) has as outcome the Cournot quantities and prices.

Capacity and price game
Conclusion

- Davidson and Deneckere (RAND, 1986) argue that the Kreps and Scheinkman result depends strongly on the chosen rationing rule.
- Madden (ET, 1998) shows, in a slightly different framework, that for uniformly elastic demands the Kreps and Scheinkman result holds, even if proportional rationing.

Conclusion

- In the standard Monopoly model we obtain the same result regardless of the choice variable of the Monopolist (price or quantity)
- This no longer holds for the Oligopoly models.
  - The equilibrium outcome depends crucially on the strategic variable.
  - Bertrand model: price.
  - Cournot model: quantity.
  - Kreps and Scheinkman model: quantity and price.
  - Flath (2012) finds out that on 70 Japanese manufacturing industries, 5 are Cournot-like, 35 are Bertrand-like, and 30 are hybrid-like.

Conclusion

- In Oligopoly models, the order of moves also plays a role.
  - Bertrand and Cournot models: simultaneous moves.
  - Stackelberg model: as Cournot (quantity competition) but sequential moves.
    - The leader selects the pair (own quantity, rival's response quantity) that maximizes its profits.
  - Kreps and Scheinkman model: sequential stages of simultaneous moves.

References

- Bertrand, J. (1883) "Book review of théorie mathématique de la richesse sociale and of recherches sur les principles mathematiques de la theorie des richesses", Journal de Savants 67: 499–508.
- Blume, A. (2003). Bertrand without fudge. Economics Letters, 78(2), 167-168.
- Cournot, A. Recherches sur les Principes Mathematiqlles de la Tleuorie des Richesses. Paris: 1838 English translation: (N. Bacon, trans.), Researches into the Mathematical Principles of the Theory of Wealth, New York: Macmillan & Company, 1897.
- Davidson, C., & Deneckere, R. (1986). Long-Run Competition in Capacity, Short-Run Competition in Price, and the Cournot Model. The RAND Journal of Economics, 17(3), 404-415.
- Flath, D. (2012). Are there any cournot industries in Japan? The Japanese Economy, 39, 3-36.

References

- Hahn, F. (1962). The Stability of the Cournot Oligopoly Solution. The Review of Economic Studies, 29(4), 329-331.
- Kartik, N. (2011). A note on undominated Bertrand equilibria. Economics Letters, 111(2), 125-126.
- Kreps, David M., and Jose A. Scheinkman. 1983. “Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes.” Bell Journal of Economics 14, no. 2 (Autumn): 326–37.
- Madden, P. (1998). Elastic demand, sunk costs and the Kreps–Scheinkman extension of the Cournot model. Economic Theory 12, 199–212
- Stackelberg, Heinrich Freiherr von (1934): Marktform und Gleichgewicht (Market Structure and Equilibrium), 2011, Translated by Bazin, Damien, Hill, Rowland, Urch, Lynn Vienna, 1934
- Tirole, Jean. 1988. The Theory of Industrial Organization. Cambridge: MIT Press.