Stochastic Calculus (Calcul Stochastique) - Solution to the Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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Answers can be formulated in English or French.

Part A : A derivative in two steps binomial tree (4 pts)

A.1) (3 pts) The NAO price of the derivative at date $t = 0$ satisfies

$$D_0 = \frac{qD_1(S_1^u) + (1-q)D_1(S_1^d)}{1+r}$$

where q denote the risk-neutral probability that satisfies

$$S_0(1+r) = qS_1^u + (1-q)S_1^d$$

From $S_1^u = uS_0$ and $S_1^d = dS_0$ we have

$$q = \frac{1+r-d}{u-d}$$

and

$$D_0 = \frac{q(auS_0+b) + (1-q)(adS_0+b)}{1+r} = \frac{qaS_0(u-d) + adS_0 + b}{1+r}$$

which is equivalent to

$$D_0 = \frac{(1+r-d)aS_0 + adS_0 + b}{1+r} = \frac{(1+r)aS_0 + b}{1+r}$$

A.2) (1 pt) When the values of the parameters are $u = 4/3$, $d = 2/3$, $S_0 = 15$, $r = 5\%$, $a = 2$ and $b = -15$, the price of the derivative is then

$$D_0 = \frac{1.05 \times 2 \times 15 - 15}{1.05} \simeq 15.71$$

Part B : Black-Scholes formula (3 pts)

We have

$$d_1 = \frac{ln\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ and } d_2 = \frac{ln\frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

with $S_0 = 15$, $K = 13$, $r = 0.08$, $\sigma = 0.22$, and $T = 0.25$. That is

$$d_1 = \frac{ln(\frac{15}{13}) + (0.08 + \frac{0.22^2}{0.25})T}{0.22\sqrt{0.25}} \simeq 1.5377 \text{ and } d_2 = \frac{ln(\frac{15}{13}) + (0.08 - \frac{0.22^2}{0.25})T}{0.22\sqrt{0.25}} \simeq 1.4277$$

and $Ke^{-rT} = 13e^{-0.08\times0.25} \simeq 12.7426$. Hence, $p = Ke^{-rT}N(-d_2) - S_0N(-d_1) = 12.7426N(-1.4277) - 15N(-1.5377)$.

Part C : Put-Call parity for American options on a dividend paying stock (13 pts)

C.1) (1 pt) The Put-Call parity at date 0 is :

$$S_0 - D - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$$

C.2) (1 pt) The arbitrage strategy at $t = 0$ consists in taking a :

- Long position in the call $(-C_0)$;
- Long position in a risk-free zero-coupon bond that costs the sum of the exercice price and the present value of the dividend $K + D$ $(-K - D)$;
- Short position in the put (P_0) ;
- Short sells a share of the underlying security (S_0) ; and
- Invest the obtained cash flow $(S_0 + P_0 - K - D - C_0)$ on the money market.

C.3) (1 pt) From $C_0 - P_0 < S_0 - D - K$, the invesment on the money market is strictly positive : $S_0 + P_0 - K - D - C_0 > 0$. So, in all cases we will obtain a positive amount $(S_0 + P_0 - K - D - C_0)e^{rT} > 0$ from the money market at date T.

C.4) (1 pt) The objective of component D in our strategy is to enable us to meet the obligation to pay any dividend generated by the underlying security that we have sold short.

C.5) (1 pt) If the put option is not exercised before maturity, the value of this portfolio at date T is :

$$C_T + De^{rT} + Ke^{rT} - P_T - S_T$$

which, when the dividend is transferred, rewrites as

$$\begin{aligned} C_T + Ke^{rT} - P_T - S_T &= max\{S_T - K; 0\} + K + K(e^{rT} - 1) - (max\{K - S_T; 0\} + S_T) \\ &= max\{S_T; K\} + K(e^{rT} - 1) - max\{K; S_T\} = K(e^{rT} - 1) \end{aligned}$$

This value is clearly positive as $rT > 0$.

C.6) (1 pt) If the put option is exercised at time $t \in [0; T)$ before the option expires, the value of this portfolio at date t is :

$$C_t + De^{rt} + Ke^{rt} - (K - S_t) - S_t = C_t + De^{rt} + K(e^{rt} - 1)$$

which, when the dividend is transferred, writes at time T as

$$C_T + K(e^{rT} - e^{r(T-t)}).$$

This value is positive as $C_T \geq 0$, $t \in [0; T)$ and $r > 0$.

C.7) (1 pt) We have shown that our portfolio has a positive value in all cases. From the positive cash flows that we obtain from the money market we have then constructed an arbitrage strategy. The non-arbitrage opportunity (NAO) then contradicts the assumption made in **C.2)**. Rather we have :

$$S_0 - D - K \leq C_0 - P_0$$

This corresponds to the lower bound of the put-call parity.

C.8) (1 pt) The arbitrage strategy at $t = 0$ consists in taking a :

— Long position in the put $(-P_0)$;

— Long position in the stock $(-S_0)$;

— Short position in the call (C_0) ;

— Sells a risk-free bond with a face value of the exercise price of the options (Ke^{-rT}) ; and

— Invest the obtained cash flow $(Ke^{-rT} + C_0 - S_0 - P_0)$ on the money market.

C.9) (1 pt) From $C_0 - P_0 > S_0 - Ke^{-rT}$, the invesment on the money market is strictly positive : $Ke^{-rT} + C_0 - S_0 - P_0 > 0$. So, in all cases we will obtain a positive amount $(Ke^{-rT} + C_0 - S_0 - P_0)e^{rT} > 0$ from the money market at date T.

C.10) (1 pt) If the call is exercised at date $t \in [0; t')$ before the dividend is paid, the value of this portfolio at date t is :

$$P_t + S_t + (K - S_t) - Ke^{-r(T-t)} = P_t + K(1 - e^{-r(T-t)})$$

which, at date T rewrites as

$$P_T + K(e^{r(T-t)} - 1)$$

This value is clearly positive as $P_T \geq 0$, $t < t' < T$ and $r > 0$.

C.11) (1 pt) If the call is exercised at date $t \in [t'; T]$ after the dividend is paid, the value of this portfolio at date t is :

$$P_t + S_t + De^{rt} - (S_t - K) - Ke^{-r(T-t)} = P_t + De^{rt} + K(1 - e^{-r(T-t)})$$

which writes at time T as

$$P_T + De^{rT} + K(e^{r(T-t)} - 1)$$

This value is positive as $P_T \geq 0$, $t \in [t'; T)$, $r > 0$ and $D \geq 0$.

C.12) (1 pt) If the call is never exercised, the value of this portfolio at date T is :

$$P_T + S_T + De^{rT} - K = max\{K - S_T; 0\} + S_T - K + De^{rT} = max\{S_T - K; 0\} + De^{rT}$$

This value is positive as $r > 0$ and $D \geq 0$.

C.13) (1 pt) We have shown that our portfolio has a positive value in all cases. From the positive cash flows that we obtain from the money market we have then constructed an arbitrage opportunity. The non-arbitrage opportunity (NAO) then contradicts the assumption made in **C.8)**. Rather we have :

$$C_0 - P_0 \leq S_0 - Ke^{-rT}$$

This corresponds to the upper bound of the put-call parity. Combining the two bounds we obtain the Put-Call parity at date 0 of **C.1)** :

$$S_0 - D - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$$