Stochastic Calculus (Calcul Stochastique) - Exam

Université Paris Dauphine-PSL - Master 1 I.E.F. (272)

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Answers can be formulated in English or French.

Part A : A derivative in two steps binomial tree (4 pts)

There are two periods, $t \in \{0, 1\}$. There are two assets. One non-risky asset (money that can be borrowed or lend) that returns $r\%$ with annual compounding at time 1. And one risky asset which is a commodity of price S_0 at time 0. At date 1, there is either an upward or a downward move. The price of the commodity is then either $S_1^u = uS_0$ or $S_1^d = dS_0$, with $u > d$. At date $t = 0$, a financial institution issues a derivative D. According to the underlying contract, the buyer of the derivative resell it at date $t = 1$ to the financial institution at price $D_1(S_1) = aS_1 + b$, where a and b are two specific numbers.

A.1) (3 pts) What is the NAO price D_0 of the derivative at date $t = 0$?

A.2) (1 pt) Compute this price for the following values of the parameters $u = 4/3$, $d = 2/3$, $S_0 = 15$, $r = 5\%$, $a = 2$ and $b = -15$.

Part B : Black-Scholes formula (3 pts)

The stock price 3 months from the expiration of an European option is £15, the exercise price of the option is £13, the risk-free interest rate is 8% per annum, and the volatility is 22% per annum.

According to the Black-Scholes formula what is the value of the European put ? (Give an answer of the form : $xN(y) + zN(t)$, where $(x, y, z, t) \in R^4$, and $N(.)$ denote the standard normal cumulative distribution function.)

Part C : Put-Call parity for American options on a dividend paying stock (13 pts)

The purpose of this exercise is to prove the Put-Call parity at date 0 for American options when the underlying security is a stock that pays a dividend. At any date $t \in \{0, ..., T\}$, let C_t and P_t denote an American call and put no arbitrage prices on the same dividend paying stock S_t with same strike K and same maturity T, with $T > 1$. The risk free interest rate with continuous compounding is denoted by $r > 0$. Assume there is a unique date t', with $0 < t' < T$, at which the dividend is paid. Let D denote the present value (at date 0) of the dividend.

C.1) (1 pt) What is the Put-Call parity at date 0 ?

C.2) (1 pt) Assume $C_0 - P_0 < S_0 - D - K$. Depict an arbitrage strategy that consists in taking a position in one unit of each option, one unit of the stock, one unit of a risk-free zero-coupon bond that costs $K + D$, and that consists in an investment of $(S_0 + P_0 - K - D - C_0)$ on the money market.

C.3) (1 pt) Is the invesment on the money market positive or negative ?

C.4) (1 pt) What is the objective of component D in your strategy ?

C.5) (1 pt) Consider the portfolio underlying your position on the financial instruments (options, stock and bond) only (so that this portfolio ignore your investment in the money market). What is the value of this portfolio at date T if the put option is not exercised before maturity ? Is this value positive or negative ?

C.6) (1 pt) What is the value of this portfolio at date T if the put option is exercised before the option expires at time $t \in [0; T)$? Is this value positive or negative ?

C.7) (1 pt) What have you shown ? What do you conclude with respect to the assumption made in **C.2)** ? Which bound of the put-call parity do you obtain ?

C.8) (1 pt) Now, assume $C_0 - P_0 > S_0 - Ke^{-rT}$. Depict an arbitrage strategy that consists in taking a position in one unit of each option, one unit of the stock, one unit of a risk-free zero-coupon bond that costs Ke^{-rT}, and that consists in an investment of $(Ke^{-rT} + C_0 - S_0 - P_0)$ on the money market.

C.9) (1 pt) Is the invesment on the money market positive or negative ?

C.10) (1 pt) Consider the portfolio underlying your position on the financial instruments (options, stock and bond) only (so that this portfolio ignore your investment in the money market). What is the value of this portfolio at date T if the call is exercised at date $t \in [0; t')$ before the dividend is paid ? Is this value positive or negative ?

C.11) (1 pt) What is the value of this portfolio at date T if the call is exercised at date $t \in [t'; T]$ after the dividend is paid ? Is this value positive or negative ?

C.12) (1 pt) What is the value of this portfolio at date T if the call is never exercised ? Is this value positive or negative ?

C.13) (1 pt) What have you shown ? What do you conclude with respect to the assumption made in **C.8)** ? Which bound of the put-call parity do you obtain ? Conclude with respect to **C.1)**.