Game Theory in Banking, Finance, and the International Arena

Master PEI - Autumn 2022

Jérôme MATHIS

www.jeromemathis.fr/PEI
password: master-PEI

LEDa - Univ. Paris-Dauphine

Chap.4 Incomplete info. games

Incomplete information games
Outline

Introduction

- In many game theoretic situations, one agent is unsure about the preferences or intentions of others.
- Incomplete information introduces additional strategic interactions and also raises questions related to “learning”.

Introduction

- Examples:
  - Bargaining
    - How much the other party is willing to pay is generally unknown to you
  - Auctions
    - How much should you be for an object that you want, knowing that others will also compete against you?
  - Market competition
    - Firms generally do not know the exact cost of their competitors
  - Signaling games
    - How should you infer the information of others from the signals they send
  - Social learning
    - How can you leverage the decisions of others in order to make better decisions

Incomplete information games
Outline

Adverse selection
Class experiment

- Play in pairs.
- One of you has a company (T) for sale. You will know the value of your company v_T.
- The other will be a potential acquirer (A).
- (A) does not know the value of the company. He only knows that T's value v_T is uniformly distributed on [0,100].
- Everyone knows that the transfer of control from (T) to (A) increases the company's value by 50% i.e. $v_A = (1.5) * v_T$.

Adverse selection
Class experiment

- Game:
  - Buyer gives a price then does not speak again
  - Seller says one word: YES or NO
  - No other communication

Adverse selection
Class experiment

- I predict that most of those whose offer was accepted will lose money.

Adverse selection
Class experiment

- What should A offer?
- When A offers p, only T's who have values v_T below p accept.
  - Expected value of company to A when T accepts is

$$1.5\frac{p}{2} = \frac{3}{4}p < p$$

- A's expected payoff is

$$\frac{3}{4}p - p = -\frac{1}{4}p$$

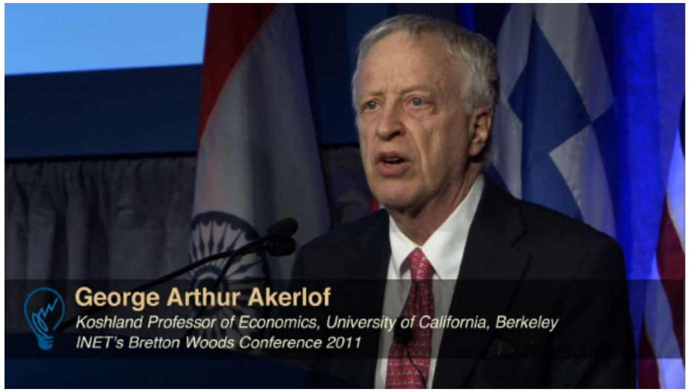
- Player A loses money if he makes an offer.

Adverse selection
Class experiment

Conclusion:

- Equilibrium price is zero
- No firms are traded
- Bad firms drive good ones out of the market

Adverse selection
George Akerlof (Nobel 2001)



Nobel Prize in Economics 2001 (with J. Stiglitz and M. Spence)

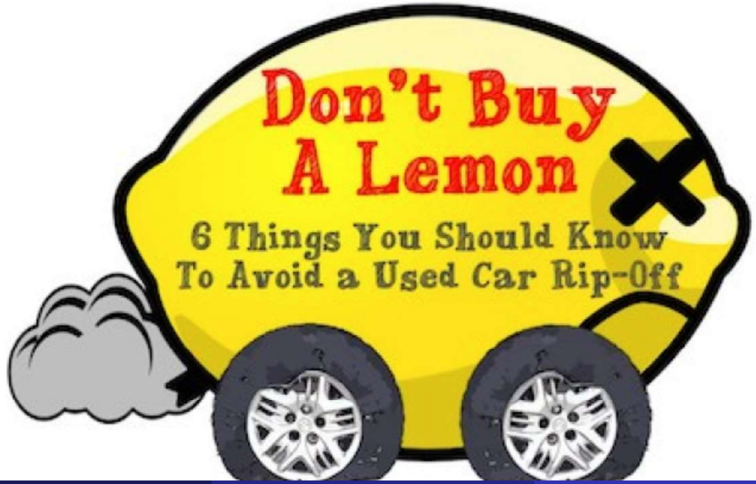
Nobel Prize motivation: For his analysys of markets with asymmetric information

Adverse selection
George Akerlof (Nobel 2001)

- **Field:** Economics of information
- **Contribution:** Studied markets where sellers of products have more information than buyers about product quality. He showed that low-quality products may squeeze out high-quality products in such markets, and that prices of high-quality products may suffer as a result.

Adverse selection
Market for used cars

George Akerlof, The Market for “Lemons”, *Quarterly Journal of Economics*, 1970.



Adverse selection
Market for used cars

- Used cars either bad “lemons” or good “peaches”
- For a lemon:
  - seller will accept 1000 euros
  - buyer will pay at most 1200 euros
- For a peach:
  - seller will accept 2000 euros
  - buyer will pay at most 2400 euros

Adverse selection
Market for used cars

- Complete information
  - Suppose that buyers can tell good cars from lemons
  - Lemons trade at price between 1000 and 1200
  - Good cars trade at price between 2000 and 2400
  - All cars are sold and final allocation is efficient
- Asymmetry of information
  - Suppose that buyers cannot tell good cars from lemons but sellers can
  - Half the cars are lemons half are good
- **Question:** Will all cars be sold? At what price(s)?

Adverse selection
Market for used cars

- Since buyers cannot distinguish bad from good cars, all cars must be sold at same price
- For sellers to be ready to sell good cars, price must be at least 2000
- However, the expected value of a car for a buyer is at most:

$$\frac{1}{2}2400 + \frac{1}{2}1200 = 1800$$

- So buyer will not be ready to offer more than 1800

Adverse selection
Market for used cars

- Can 1800 be equilibrium price?
  - No: at that price only bad cars are sold and buyer is not ready to offer more than 1200
- **Only equilibrium is such that:**
  - Only bad cars are sold at a price between 1000 and 1200
  - Good cars are driven out of the market

Adverse selection
Market for used cars

- More generally, let l be the fraction of lemons and $1 - l$ the fraction of peaches
- If both types are traded, the average value of a car to the buyer is: $EVB = (1200)l + (2400)(1 - l)$
- The peach owner only sells his car if the price is above 2,000
- So, peaches are traded only if $(1200)l + (2400)(1 - l) > 2000$, i.e. if $l < 1/3$
- There must be at least 2/3 of peaches for them to be traded

Adverse selection
Market for used cars

- **The problem comes from asymmetric information, not uncertainty**.
- If both sellers and buyers were uncertain about value:
  - Expected value for a buyer is
$$\frac{1}{2}1200 + \frac{1}{2}2400 = 1800$$
  - Expected value for a seller is
$$\frac{1}{2}1000 + \frac{1}{2}2000 = 1500$$
  - Cars are sold at a price between 1500 and 1800.

Incomplete information games
Outline

Games of incomplete information

- Up until this point, all games were with complete information: knew everything about the point of view of the other players
- Most practical situations, uncertainty about some aspect of the payoffs of other players: for example his cost of action, his discount factor, the value of the car...

Games of incomplete information
Example: game played in class

- The game we played in class is indeed a strategic game
- The difference with games we saw before is that all players do not have access to the same information
- The uninformed buyer moves first and in equilibrium should take into account the expected behavior of the informed seller who moves second
- The equilibrium strategy of the seller is: accept any price above his valuation
- Given this expected strategy of the buyer, we showed that optimal for the buyer not to make an offer

Games of incomplete information

- Of course in the case of the used car markets you can imagine that the seller could offer a warranty when selling the car
- The level of the warranty would reveal some of the private information
- We see a key difference between simultaneous games and dynamic games
- In dynamic games beliefs about the information held by other players evolve during the course of the game

Games of incomplete information
Roadmap

- We first look at the simultaneous games with incomplete information
- Then we look at extensive form games with incomplete information.
- Two classes of applications:
  1. signalling games
  2. models of reputation

Games of incomplete information
Complete information games

A complete information simultaneous game is described by:

- A finite set of players $N = \{1, 2, ..., n\}$
- Strategy sets $S_1, ..., S_n$
- Payoff functions $u_i : S_1 \times ... \times S_n \longmapsto \mathbb{R}$ for each $i \in N$

Games of incomplete information
Incomplete information games

Incomplete information games: additional element is the information each player has, what we call his "type".

- A finite set of players $N = \{1, 2, ..., n\}$
- A set of types profile $\Theta \equiv \Theta_1 \times ... \times \Theta_n$, where Θ_i is the set of possible types for player i
- A joint probability distribution $p(\theta_1, ..., \theta_n)$ over types
- A set of strategies profile $S \equiv S_1 \times ... \times S_n$, where S_i denotes player i's strategy set
- Payoff function $u_i : S \times \Theta \longmapsto \mathbb{R}$

Games of incomplete information
Example

- **In the game we played in class,** the buyer has no private information while the type of the seller is $\Theta_S = [0, 100]$.
  - It is a game of incomplete information on one side.
- Some two players game are of incomplete information on both side.

Example (Public good problem)

	Call	*Don't*
Call	$(1 - c_1, 1 - c_2)$	$(1 - c_1, 1)$
Don't	$(1, 1 - c_2)$	$(0, 0)$

Games of incomplete information
Example

Example (Public good problem)

	Call	*Don't*
Call	$(1 - c_1, 1 - c_2)$	$(1 - c_1, 1)$
Don't	$(1, 1 - c_2)$	$(0, 0)$

- In this game
  - Both players are informed about their own cost
  - Believe that cost of other player is drawn from a uniform distribution on $[\underline{c}, \overline{c}]$
  - In this case $\Theta_1 = \Theta_2 = [\underline{c}, \overline{c}]$

Games of incomplete information
Strategies

- Strategy is now an association between a type and an action
- Means that depending on your private information, you might choose different things
- For instance in the game played in class, if value of company (seller's type) is 50, the strategy of the seller is to accept any offer above 50.
- If it is 40, strategy is different, the strategy of the seller is to accept any offer above 40

Games of incomplete information
Solution concept

- We just consider the generalization of the Nash equilibrium concept
- It is a set of strategies such that if the other players play their Nash equilibrium strategies, you do not want to change your choice
- Difference is that you are uncertain about the other player's information, so to judge whether your choice is indeed the best, you compute the expected payoff given the probability of the others types
- Call the concept *Bayesian Nash equilibrium*

Games of incomplete information
Solution concept

Definition

A **Bayesian (mixed) strategy** for player i is a function

$$\mu_i : \Theta_i \longmapsto \Delta(S_i)$$

Definition

A Bayesian strategy profile $(\mu_1, ..., \mu_n)$ is a **Bayesian Nash Equilibrium** if for all i, θ_i and μ_i' we have

$$\mathbb{E}_{\mu}\left[u_i(.)|\theta_i\right] \geq \mathbb{E}_{\mu_i', \mu_{-i}}\left[u_i(.)|\theta_i\right]$$

Games of incomplete information
Solution concept

- The expected payoff $\mathbb{E}_{\mu}\left[u_i(.)|\theta_i\right]$ writes as

$$\sum_{\theta_{-i}\in\Theta_{-i}} \mathbb{P}(\theta_{-i}|\theta_i) \sum_{s\in S} \mu(s|\theta_i, \theta_{-i}) u_i(s; \theta_i, \theta_{-i})$$

when the sets of types Θ_i are finite, otherwise writes as

$$\int_{\theta_{-i}\in\Theta_{-i}} \sum_{s\in S} \mu(s|\theta_i, \theta_{-i}) u_i(s; \theta_i, \theta_{-i}) d\mathbb{P}(\theta_{-i}|\theta_i)$$

with $\mu(s|\theta) \equiv \mu_1(s_1|\theta_1) \times \mu_2(s_2|\theta_2) \times ... \times \mu_n(s_n|\theta_n)$.

Incomplete information games
Outline

Solving Bayesian games
Public good problem

Example (Public good problem)

	Call	*Don′t*
Call	$(1-c_1, 1-c_2)$	$(1-c_1, 1)$
Don′t	$(1, 1-c_2)$	$(0, 0)$

- Let us solve this game in a special case where:
  - Player 1 has known cost $c_1 = 1/4$
  - Player 2 has cost $c_2 = 1/2$ with probability p and $c_2 = 3/2$ with probability $1-p$
  - More likely to have a high cost: $p < 1/2$.

Solving Bayesian games
Public good problem

Example (Public good problem)

$c_2 = \frac{1}{2}$	C	D
C	$(\frac{3}{4}, \frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, \frac{1}{2})$	$(0, 0)$

$c_2 = \frac{3}{2}$	C	D
C	$(\frac{3}{4}, -\frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, -\frac{1}{2})$	$(0, 0)$

- Strategies in this environment can be defined as follows:
  - Choice of player 1: $\mu_1(c_1) = \mu_1$
  - Choice of player 2 depending on his cost: $\mu_2(c_2)$
    - A priori could have a different choice depending on costs, i.e could have $\mu_2(c_2 = \frac{1}{2}) \neq \mu_2(c_2 = \frac{3}{2})$.

Solving Bayesian games
Public good problem

- What is the set of Bayesian Nash Equilibrium?

Proposition

Unique Bayesian Nash Equilibrium is for player 1 to play “Call” and for player 2 to play “Don′t” for all c_2.

- In this example it turns out that in equilibrium the strategy of player 2 does not depend on his information, both types do the same
- Clear that it is a Nash equilibrium. We show below that it is unique.

Example (Public good problem)

$c_2 = \frac{1}{2}$	C	D
C	$(\frac{3}{4}, \frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, \frac{1}{2})$	$(0, 0)$

$c_2 = \frac{3}{2}$	C	D
C	$(\frac{3}{4}, -\frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, -\frac{1}{2})$	$(0, 0)$

Proof.

Let us solve it for mixed strategy. Let $\mu^i(C|c_i)$ denote the probability that player i plays C if c_i.

$$\begin{aligned}
\pi^1(C) &= \frac{3}{4} \\
\pi^1(D) &= p\mu^2(C|c_2 = 1/2) + (1-p)\mu^2(C|c_2 = 3/2) \\
\text{So, } \mu^1(C) &= \begin{cases} 1 \text{ if } \frac{3}{4} > p\mu^2(C|c_2 = \frac{1}{2}) + (1-p)\mu^2(C|c_2 = \frac{3}{2}) \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

Example (Public good problem)

$c_2 = \frac{1}{2}$	C	D
C	$(\frac{3}{4}, \frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, \frac{1}{2})$	$(0, 0)$

$c_2 = \frac{3}{2}$	C	D
C	$(\frac{3}{4}, -\frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, -\frac{1}{2})$	$(0, 0)$

Proof.

$$\begin{aligned}
\pi^2\left(C \middle| c_2 = \frac{3}{2}\right) &= -\frac{1}{2} < \pi^2\left(D \middle| c_2 = \frac{3}{2}\right) \\
\text{So, } \mu^2\left(C \middle| c_2 = \frac{3}{2}\right) &= 0 \\
\pi^2\left(C \middle| c_2 = \frac{1}{2}\right) &= \frac{1}{2} \text{ and } \pi^2\left(D \middle| c_2 = \frac{1}{2}\right) = \mu^1(C)
\end{aligned}$$

□

Example (Public good problem)

$c_2 = \frac{1}{2}$	C	D
C	$(\frac{3}{4}, \frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, \frac{1}{2})$	$(0, 0)$

$c_2 = \frac{3}{2}$	C	D
C	$(\frac{3}{4}, -\frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, -\frac{1}{2})$	$(0, 0)$

Proof.

$$\text{Now, } \mu^1(C) = \mathbf{1}_{\{\frac{3}{4} > p\mu^2(C|c_2 = \frac{1}{2}) + (1-p)\times 0\}} = \mathbf{1}_{\{p\mu^2(C|c_2 = \frac{1}{2}) < \frac{3}{4}\}} = 1$$

because $p\mu^2(C|c_2 = \frac{1}{2}) \leq p < \frac{1}{2} < \frac{3}{4}$.

$$\text{So, } \pi^2\left(C \middle| c_2 = \frac{1}{2}\right) = \frac{1}{2} < 1 = \pi^2\left(D \middle| c_2 = \frac{1}{2}\right) = \mu^1(C)$$

□

Example (Public good problem)

$c_2 = \frac{1}{2}$	C	D
C	$(\frac{3}{4}, \frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, \frac{1}{2})$	$(0, 0)$

$c_2 = \frac{3}{2}$	C	D
C	$(\frac{3}{4}, -\frac{1}{2})$	$(\frac{3}{4}, 1)$
D	$(1, -\frac{1}{2})$	$(0, 0)$

Proof.

$$\text{and, } \mu^2\left(C \middle| c_2 = \frac{1}{2}\right) = 0.$$

Hence, $\mu^2(C|c_2) = 0$ for every c_2.

So, $\mu^1(C) = 1$.

Therefore (C, D) is the unique Nash equilibrium. □

Solving Bayesian games
Public good problem: version 2

- Suppose now c1 and c2 drawn from uniform distribution on $[0,2]$

Proposition

The unique Bayesian Nash equilibrium is to play the same strategy for both players.

This strategy consists in playing "Call" if $c_i \leq \frac{2}{3}$ and playing "Don't" otherwise.

Solving Bayesian games
Public good problem: version 2

Proof.

$$\pi^1(C|c_1) = 1 - c_1 \text{ and } \pi^1(D|c_1) = \int_{c_2} \mu^2(C|c_2)d\mathbb{P}(c_2)$$

So

$$\begin{aligned} \pi^1(C|c_1) &\geq \pi^1(D|c_1) \iff 1 - c_1 \geq \int_{c_2} \mu^2(C|c_2)d\mathbb{P}(c_2) \\ &\iff c_1 \leq 1 - \int_{c_2} \mu^2(C|c_2)d\mathbb{P}(c_2) \equiv c_1^* \end{aligned}$$

and

$$\mu^1(C|c_1) = \begin{cases} 1 \text{ if } c_1 \leq c_1^* \\ 0 \text{ otherwise} \end{cases}.$$

□

Solving Bayesian games
Public good problem: version 2

Proof.

Similarly,

$$\mu^2(C|c_2) = \begin{cases} 1 \text{ if } c_2 \leq c_2^* \\ 0 \text{ otherwise} \end{cases}$$

with $c_2^* \equiv 1 - \int_{c_1} \mu^1(C|c_1)d\mathbb{P}(c_1) = 1 - \int_0^{c_1^*} d\mathbb{P}(c_1) = 1 - \frac{1}{2}c_1^*$.

From

$$\begin{cases} c_2^* = 1 - \frac{1}{2}c_1^* \\ c_1^* = 1 - \frac{1}{2}c_2^* \end{cases}$$

we get $c_1^* = 1 - \frac{1}{2}\left(1 - \frac{1}{2}c_1^*\right) = \frac{1}{2} + \frac{c_1^*}{4} \iff \frac{3}{4}c_1^* = \frac{1}{2} \iff c_1^* = \frac{2}{3}$.

Therefore

$$\mu^i(C|c_i) = \begin{cases} 1 \text{ if } c_i \leq \frac{2}{3} \\ 0 \text{ otherwise} \end{cases}.$$

□

Incomplete information games
Outline

Dynamics
Incomplete information games

- In dynamic games of incomplete information, choices by informed players can reveal information.
- We need to keep track of the evolution of beliefs during the game.
- Equilibrium is defined by:
  - strategies and beliefs of players
  - each player maximizes his payoff given his beliefs and the equilibrium strategies of the other players
  - beliefs are updated given choices observed and equilibrium strategies of the different players

Dynamics
Updating beliefs

- Example of used car market:
  - Suppose the seller moves first and can offer a warranty
  - Given cost of warranty, possible that in equilibrium only sellers of good cars offer a warranty
  - Following an offer of warranty, you update your beliefs and put a probability one that the car is good
  - Notice that the updating is done given strategy of seller
  - Updating can be much more complicated in general

Dynamics
Used cars

- Seller of car knows quality of the car
- Sequential game with two steps:
  - Step 1: seller chooses strategy W (offer a warranty) or N (no warranty) and chooses a price
  - Step 2: buyer chooses to buy car or not
- Assumption about payoffs:
  - For a good car, buyer ready to pay 2400, it is worth 2000 for seller and warranty is expected to cost 250 for seller
  - For a bad car, buyer ready to pay 1200, it is worth 1000 for seller and warranty expected to 1500 for seller

Dynamics
Used cars

- There is an equilibrium where:
- Strategies are:
  - Seller of good car chooses W and price 2400
  - Seller of bad car chooses N and price 1200
  - If seller chose W, buyer buys if price is less than 2400
  - If seller chose N, buyer buys if price is less than 1200
- Beliefs are:
  - If seller chooses W he has a good car
  - If chooses N he has a bad car

Dynamics
Used cars

- Need to check:
  - Equilibrium strategy of each player maximizes payoff given other player's equilibrium strategies
  - Equilibrium beliefs are consistent with equilibrium strategies

Dynamics
Used cars

Step 1: does the seller behave optimally given equilibrium beliefs and strategies of buyers?

- Seller of a good car
  - If he sells with no warranty, given equilibrium strategy and belief of buyer, highest price he can offer is 1200. He can get a maximum of: $1200 - 2000 = -800$
  - If he sells with warranty, gets in equilibrium: $2400 - 2000 - 250 = 150$
- Seller of bad car:
  - If he sells with no warranty, gets: $1200 - 1000 = 200$
  - If he sells with warranty, gets: $2400 - 1000 - 1500 = -100$.

Dynamics
Used cars

Step 2: does the buyer behave optimally given his equilibrium beliefs and equilibrium strategies of sellers?

- Obviously the case:
  - If seller chose W, given that the equilibrium belief is then that the car is good for sure, best strategy is to accept at any price less than 2400
  - If seller chose N, given that the equilibrium belief is then that the car is bad for sure, best strategy is to accept at any price less than 1200.

Dynamics
Used cars

Step 3: Are equilibrium beliefs consistent with equilibrium strategies?

- Obviously the case:
  - In equilibrium only sellers of good cars offer a warranty, so after observing a seller offering a warranty, assigning a probability of one that he has a good car is consistent
  - Same for bad cars
- Updating could be more complicated in general (need to apply Baye's rule)...

Dynamics
Used cars

Conclusion:

- Warranty serves as a costly signal
- Signal needs to be more costly for bad types than for good types for the signalling to work
- Lots of applications: education, politics, biology...

Incomplete information games
Outline

Education as signalling

- Two types of employees: high quality H and low quality low L
- Potential employers in high paying sector are ready to pay $\$160,000$ for H and $\$60,000$ for L
- Employer does not observe the type of employee before hiring but observes the education history
- Private information of employee is his quality

Education as signalling

Education is costly:

- Tolerance for a tough course varies by individuals:
  - for H, cost of taking a tough one year course is equivalent to $\$12,500$
  - for L, cost of taking a tough one year course is equivalent to $\$22,300$
- Employer can base his hiring policy on the number n of tough courses taken

Education as signalling

Two types of equilibrium can exist:

1. Seperating equilibria: the H quality take more tough courses than the low and can thus signal their ability and get a higher pay.
2. Pooling equilibria: both types choose the same level of education and get the same pay.

Education as signalling

For a seperating equilibria to exist, the number *n* of tough courses the employer expects must be such that:

1. The high type prefer taking *n* courses and getting the high pay than taking no course and taking the low
2. The low type prefer not taking courses and getting the low pay than taking *n* courses and getting high pay

- Conditions are:

$$\begin{aligned} 160,000 - 12,500n &\geq 60000, \text{ i.e., } n \leq 8 \\ 60,000 &\geq 160,000 - 22,300n, \text{ i.e., } n \geq 4.48 \end{aligned}$$

Education as signalling

Following is an equilibrium:

- Types H take 5 tough courses
- Types L do not take tough courses
- If the employer sees strictly less than 5 tough courses, he believes the type is L with probability 1 and pays $60,000
- If the employer sees more than 5 tough courses, he believes the type is H with probability 1 and pays $160,000

Incomplete information games
Outline

Cheap talk

- Lot of situations where more informed parties communicate information to less informed.
- Sometimes people say that the information communicated to them is just “cheap talk”.
- This statement is really an equilibrium statement.

Cheap talk

Consider the following environment:

- Financial adviser can sell PEL or Assurance Vie.
- What is better for you depends on your consumption profile, your future prospects... what we will call your type.
- For type A, PEL gives a payoff of 2 and Assurance Vie a payoff of 1.
- For type B, PEL gives a payoff of 1 and Assurance Vie a payoff of 2.
- Financial adviser knows your type but you don’t.

Cheap talk

Game has the following timing:

- Financial advisor sends a message: buy A or buy B
- Client takes a decision A or B

Cheap talk
Perfectly aligned interests

Suppose first that the financial advisor has interest perfectly aligned with yours

- For instance could be the case that he wants to keep you as a client and therefore adopts completely your perspective
- In that case resolution of the game is clear: financial advisor tells the truth and client follows the advice

Cheap talk
Non aligned interests

In general the financial advisor will receive a commission. Suppose the client is told that the advisor gets a commission X if he sells Assurance vie.

- Is the following an equilibrium?
  - Strategy of advisor is tell the truth
  - Strategy of client is to follow advice
  - Belief of client is that the advisor is telling the truth

Cheap talk

For this to be an equilibrium, need:

1. Advisor wants to tell truth given beliefs and strategies of client
2. Client wants to follow advice given beliefs and strategy of advisor
3. Beliefs are consistent with equilibrium play

- It is an equilibrium only if X is not too high!

Cheap talk

- If interests not too misaligned, can still communicate some information
- There is a conflict between incentives given to advisor and quality of his advice
- Applications of these types of model to analyzing lobbying for instance

Incomplete information games
Outline

Correcting false positive clinical test result
Updating

- In the previous examples updating of beliefs was simple: after the informed party played you were either completely informed about his information or were uninformed as before
- Warranty: either there is a seperating equilibrium and then you know for sure that the warranty signals a good car, or there is none and you can't learn anything from what the seller does
- We are now going to see example where there is partial learning
- Updating will be done according to Baye's rule

Correcting false positive clinical test result
Updating

- Suppose that you have just received a test result indicating that you have a rare disease.
  - Unfortunately, the disease is life-threatening, but you have some hope because the test is capable of producing "false positives," and the disease is rare.
  - Your doctor tells you that the test is fairly accurate, with a false positive rate of only 1 percent.
  - The rate of the disease for those in your socio-economic group is only one per 10,000.
- What are your chances of having the disease?
  - Write down your estimate of the likelihood of having the rare disease, given a positive test result.

Correcting false positive clinical test result
Updating

- Confronted with this problem, most people conclude will that it is more likely than not that the person actually has the disease, but such a guess would be seriously incorrect.
- The 1 percent false positive rate means that testing 10,000 randomly selected people will generate about 100 positive results (1%), but on average only one person out of 10,000 actually has the disease.
- Thus the chances of having the disease are only less than one in a hundred, even after you have tested positive with a test that is correct 99 times out of 100.

Correcting false positive clinical test result
Baye's rule

- Suppose a variable X can take two values x_1 and x_2 and a variable Y can also take two values y_1 and y_2.
- If these two variables are linked somehow, observing value of Y can inform you on X.
- We use Baye's rule for updating of probabilities:

$$\Pr[x_1|y_1] = \frac{\Pr[x_1 \cap y_1]}{\Pr[y_1]} = \frac{\Pr[y_1|x_1]\Pr[x_1]}{\Pr[y_1|x_1]\Pr[x_1] + \Pr[y_1|x_2]\Pr[x_2]}$$

Correcting false positive clinical test result
Baye's rule

Let D denotes "Disease", $\bar{D}$ denotes "Not Disease" we have:

$$\begin{aligned}\Pr[D|Test+] &= \frac{\Pr[Test+|D]\Pr[D]}{\Pr[Test+|D]\Pr[D]+\Pr[Test+|\bar{D}]\Pr[\bar{D}]} \\ &= \left(1+\frac{\Pr[Test+|\bar{D}]\Pr[\bar{D}]}{\Pr[Test+|D]\Pr[D]}\right)^{-1}\end{aligned}$$

with

$$\Pr[Test+|\bar{D}] = 0.01 \qquad \Pr[\bar{D}] = 0.9999$$

$$\Pr[Test+|D] = 0.99 \qquad \Pr[D] = 0.0001$$

Correcting false positive clinical test result
Baye's rule

Therefore

$$\begin{aligned}\Pr[D|Test+] &= \frac{0.99\times 0.0001}{0.99\times 0.0001+0.01*0.9999} \\ &= \frac{1}{102}\simeq 0.0098039 < 1\%.\end{aligned}$$

Evidence from a simplified poker
Game

Game designed to represent the decision to bluff in poker:

- Deck of cards with 4 aces and 4 kings
- Game starts with everyone putting 1 $ on the table
- One player, called the informed player, draws a card from the deck
- Decides to raise, by putting another dollar or fold (in which case the uninformed gets initial stake)
- If informed decides to raise, uninformed needs to decide whether to fold or to call by putting another dollar
- If he calls, the card is revealed: informed wins if the card is an ace and uninformed wins if it is a king

Evidence from a simplified poker
Game

Game designed to represent the decision to bluff in poker:

- If informed draws an ace, dominant strategy to raise
- If informed draws a king more subtle
  - Can you have an equilibrium where informed always raises and uninformed always calls?
  - Can you have an equilibrium where informed never raises when he has a king and the uninformed never calls?

Evidence from a simplified poker
Game

- The equilibrium is necessarily such that if he gets a king, the informed player sometimes raises and sometimes doesn't (mixed strategy) and the uninformed folds with some probability as well
  - bluff rate denoted by β: probability with which informed raises when he gets a King
  - call rate is denoted γ: probability with which uninformed chooses call when informed raises
- We have:
  - An informed who raises on a king, looses 2 with probability γ and wins 1 with probability $1-\gamma$.
  - If he folds, looses 1 for sure
  - In a mixed strategy, he is indifferent:

$$\begin{aligned} -2\gamma + 1 * (1-\gamma) &= -1 \\ \iff & \gamma = 2/3 \end{aligned}$$

Evidence from a simplified poker
Game

- For uninformed, derivation is more complex: need to calculate, the belief he has that an informed player who raises has an ace.
- Formula used is Baye's rule

$$\begin{aligned} \Pr[Ace|Raise] &= \frac{\Pr[Raise|A]\Pr[A]}{\Pr[Raise|A]\Pr[A] + \Pr[Raise|K]\Pr[K]} \\ &= \frac{1*\frac{1}{2}}{1*\frac{1}{2}+\beta*\frac{1}{2}} \\ &= \frac{1}{1+\beta} \end{aligned}$$

Evidence from a simplified poker
Game

- For uninformed to be mixing between fold and call, needs to be indifferent between the two.
  - If he folds, looses 1
  - If he raises, looses 2 with probability $\frac{1}{1+\beta}$ and win 1 with probability $1-\frac{1}{1+\beta}$.
- Indifference gives:

$$\begin{aligned} -2\frac{1}{1+\beta} + 2*(1-\frac{1}{1+\beta}) &= -1 \\ \iff & \beta = 1/3 \end{aligned}$$

Evidence from a simplified poker
Game

Equilibrium is such that:

- Informed player who gets an ace always raises
- Informed player who gets a king, raises with probability $1/3$
- An uninformed player calls with probability $2/3$

Evidence from a simplified poker
Game