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Elastic demand, sunk costs and the Kreps–Scheinkman extension of the Cournot model★

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Summary. The paper shows that, with any rationing mechanism between the efficient and proportional extremes, the Kreps-Scheinkman two-stage quantity-price game reduces to the Cournot model if demand is uniformly elastic and if all costs are sunk at the first stage, thus providing positive results to set against existing negative statements.

JEL Classification Numbers: C72, D43, L13.

1 Introduction

Kreps and Scheinkman (1983) have shown how the classic Cournot outcome can emerge from a two-stage game where quantities are simultaneously chosen at the first stage, followed at stage two by simultaneous price announcements with demands rationed via the "efficient" rationing rule. Davidson and Deneckere (1986) argued that this result is not robust – the Cournot outcome is not an equilibrium if rationing follows the opposite ("proportional") extreme to the efficient rule or any rule "strictly between" these two extremes. The Davidson and Deneckere proof assumes zero costs at the first stage, but the conclusion remains if these costs are sufficiently small and if the rationing scheme is sufficiently different from the efficient rule (see Tirole, (1988, pp. 212–218, 222–233); see also the discussions of Dixon, 1987, p. 269; Vives, 1993, pp. 466–467).

The objective of this note is to present a family of two-stage models à la Kreps–Scheinkman in which the Davidson-Deneckere problem disappears. Our critical assumptions are (a) that demand is uniformly elastic, a possibility ruled out in the earlier papers, and (b) that all costs are incurred at the first stage, a restriction not needed in the earlier papers. Assumption (a) is

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easily satisfied in models based on CES utility, such as those of the monopolistic competition literature inspired by Dixit and Stiglitz (1977). Assumption (b) requires that all costs (e.g. capacity and output) are sunk prior to the output being brought to market and the pricing decision made. It is an empirical matter as to whether our assumption or the Kreps–Scheinkman (1983, p. 337) alternative (capacity chosen at stage one, output chosen after prices and demands are realised at stage two) is the more appropriate. However within our two assumptions our model is quite general – there is an arbitrary number of firms with arbitrary cost functions, and an arbitrary rationing scheme between (and including) the efficient and proportional extremes. The result is a strong equivalence between the Cournot outcome and perfect equilibrium of the two-stage game. In Tirole's (1988) terminology our two-stage Kreps–Scheinkman model has *the exact Cournot reduced form*; following any quantities chosen at the first stage, Nash equilibrium in the second stage pricing subgame induces unique expected payoffs which are the Cournot payoffs corresponding to the quantities, ensuring in particular that perfect equilibria of the two stage game coincide with Cournot-Nash equilibrium.

We should point out that this proposition is not that surprising, particularly in the light of Allen and Hellwig (1986), whose results imply that under proportional rationing with our demand assumptions, the unique expected payoffs in a Bertrand-Edgeworth equilibrium (= our second stage pricing subgame) are those induced by "market-clearing" prices if the given (first stage in our case) aggregate quantity is at a point on the demand curve where demand is elastic. Since proportional rationing provides the greatest incentive to upward price deviation, the survival of the Allen-Hellwig result to general rationing schemes is to be expected, and indeed is demonstrated here. We show also that this strong Cournot/Kreps–Scheinkman equivalence is consistent with existence and uniqueness of Cournot-Nash equilibrium, under relatively mild additional assumptions which allow uniform unit elastic demand as an extreme. Moreover a slightly weaker equivalence holds either if the range of output levels over which demand is elastic is sufficiently wide, or if costs are sufficiently large.

The elastic demand and sunk cost assumptions used to generate the equivalence results are restrictive assumptions, particularly when taken together. What follows may therefore be interpreted as negative towards the extent of relevance of the Cournot outcome.

On the other hand, within these two assumptions, the paper provides strong and robust support for the Cournot story.

2 Definitions and assumptions

There are n firms, with firm i producing a quantity of a homogeneous good $q_i \geq 0$ at cost $c_i(q_i), i = 1, \dots, n$, where $c_i : R_+ \to R_+$ is firm i's cost function and $c_i(0) = 0, i = 1, \dots, n$. Given our sunk cost assumption there is no need to distinguish capacity and output decisions. Market demand when there is a single price p is $Q = D(p)$, and market revenue in terms of price is then $\phi(p) = p\, D(p)$.

Assumption 1. (a) The demand function $D : R_{++} \to R_{++}$ is C^2 with $D'(p) < 0$ everywhere, $\lim_{p \to 0} D(p) = +\infty$ and $\lim_{p \to \infty} D(p) = 0$
(b) There exists $a \geq 0$ such that the market revenue function $\phi : R_{++} \to R_{++}$ is strictly increasing on $(0, a)$ and non-increasing on (a, ∞).

Part (a) of assumption 1 insists that the market demand curve is well-behaved, downward sloping and asymptoting to the axes. Part (b) says that market demand is inelastic at prices $p < a$, and elastic at $p > a$. The case $a = 0$ then indicates *uniformly elastic demand*, while $a > 0$ admits *eventually inelastic demand.*

Assumption 1 may be equivalently and alternatively stated in terms of the inverse market demand, $p = F(Q)$ and market revenue in terms of quantity, $\psi(Q) = QF(Q)$, as follows.

Assumption 2. (a) The inverse market demand function $F : R_{++} \to R_{++}$ is C^2 with $F'(Q) < 0$ everywhere, $\lim_{Q \to \infty} F(Q) = 0$ and $\lim_{Q \to 0} F(Q) = +\infty$
(b) There exists $b \geq 0$ such that the market revenue function $\psi : R_{++} \to R_{++}$ is non-decreasing on $(0, b)$ and strictly decreasing on (b, ∞)

The equivalence between assumptions 1 and 2 is straightforward to demonstrate, with $F = D^{-1}$ and $b = D(a); b = +\infty$ is then uniformly elastic demand.

A well-known special case of the uniform elastic demand specification is provided by the constant elasticity demand functions where, for example, $D(p) = p^{-\varepsilon}, \varepsilon \geq 1$; then $\phi(p) = p^{1-\varepsilon}, F(Q) = Q^{-\frac{1}{\varepsilon}}$ and $\psi(Q) = Q^{1-\frac{1}{\varepsilon}}$ Notice that assumption 1/2 does not entail necessarily that $\lim_{p \to \infty} \phi(p)$ $(= \lim_{Q \to 0} \psi(Q)) = 0$; for instance, the constant unit elastic example above $(\varepsilon = 1)$ satisfies assumption 1/2 but $\lim_{p \to \infty} \phi(p) = \lim_{Q \to 0} \psi(Q) = 1$. We remark here that the specification of this paper has been stretched to accommodate uniform unit elasticity, since common examples give rise to this case. For instance, Cobb-Douglas utility functions generate unit elastic individual demands, and overlapping generation models with consumption only in old age and income only when young produce unit elastic aggregate demand. Nevertheless we shall assume, as seems most natural and plausible, that firms earn zero revenue if $Q = 0$, which will allow a payoff discontinuity in our Cournot and Kreps–Scheinkman models.

In the *Cournot model* firms choose output levels $q_i, i = 1, ..., n$ simultaneously, producing an aggregate output $Q = \sum_{j=1}^{n} q_j$; in the sequel Q will always denote this aggregate output and q denotes the vector $(q_1, ..., q_n)$. Cournot payoff functions are $\pi_i^c : R_+^n \to R$, $i = 1, \ldots, n$ where;

$$\pi_i^c(q_1,\dots,q_n) = \begin{cases} q_1\ F(Q) - c_i(q_i) & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0 \end{cases}$$

The separate description for $q_i = 0$ here is needed only when $Q = 0$; notice again the possibility of a discontinuity in π_i^c at $Q = 0$ (e.g. with unit elastic demand).

In the *Kreps–Scheinkman model* firms choose output levels $q_i, i = 1, ..., n$ simultaneously at stage 1. Then, with production costs sunk and with production levels common knowledge, firms choose prices $p_i > 0, i = 1, \dots, n$ simultaneously at stage 2; p denotes the vector $(p_1, \dots, p_n)$. In Kreps and Scheinkman (1983), and in Osborne and Pitchik (1986) and Vives (1986), demand at stage 2 is rationed amongst firms according to the so-called efficient (or surplus-maximizing) rule; the following is the demand faced by firm i following the production vector q if the announced, stage 2 prices are p;

$$\Delta_{iE}(q,p) = \max\left\{0, \left[D(p_i) - \sum_{p_k < p_i} q_k\right] \cdot \frac{q_i}{\sum_{p_k = p_i} q_k}\right\}$$

Here firms charging less than firm i serve those consumers with the highest valuation of the good, leaving the square bracket term to be shared amongst firms charging p_i in proportion to their production levels.

At an opposite extreme is the proportional (or Beckmann, 1965) rule, used in particular by Allen and Hellwig (1986);

$$\Delta_{iP}(q,p) = \max\left\{0, \left[1 - \sum_{p_k<p_i} \frac{q_k}{D(p_k)}\right] \cdot D(p_i) \cdot \frac{q_i}{\sum_{p_k=p_i} q_k}\right\}$$

Here the consumers served by lower priced firms are chosen randomly; $q_k/D(p_k)$ is then the fraction of consumers served by k, thus leaving the square bracket fraction to be served by firms charging p_i.

Since D is decreasing it follows that $\sum_{p_k<p_i} (q_k/D(p_k)) \cdot (D(p_i)) \leq \sum_{p_k<p_i} q_k$, and so $\Delta_{iE}(q,p) \leq \Delta_{iP}(q,p)$ for all $q \in R_+^n, p \in R_{++}^n$. As in Davidson and Deneckere (1986), our rationing specification allows the mechanism to take on forms intermediate between the efficient and proportional extremes. Let $\Delta_i(q,p)$ denote the stage 2 demand facing i after production q when prices p are announced. We assume

Assumption 3. The rationed demand function at stage 2 of the Kreps–Scheinkman game for firm $i, i = 1, \dots, n$ is $\Delta_i : R_+^n \times R_{++}^n \to R_+$ and satisfies;

(i) $\Delta_{iE}(q,p) \leq \Delta_i(q,p) \leq \Delta_{iP}(q,p), (q,p) \in R_+^n \times R_{++}^n$
(ii) Δ_i depends only on these p_i for which $q_i > 0$.

At stage 2 revenues accrue to the firms, firm i receiving $R_i(q,p)$ where;

$$R_i(q,p) = p_i \min[q_i, \Delta_i(q,p)], \quad i = 1, \ldots, n$$

Thus payoffs in the full two-stage Kreps–Scheinkman game are $\pi_i^{KS} : R_+^n \times R_{++}^n \to R$ where, for $i = 1, \ldots, n$;

$$\pi_i^{KS}(q,p) = R_i(q,p) - c_i(q_i).$$

3 Equivalence of Kreps–Scheinkman and Cournot under elastic demand

The broad issue is whether the Cournot model can "survive" in some sense the addition of the second stage endogenous pricing subgame, thus removing the need to invoke the auctioneer to justify the emergence of market-clearing prices. A strong sense of "survival" in this context occurs when, in Tirole's (1988) terminology, the Kreps–Scheinkman model has the exact Cournot reduced form, meaning that if $q \in R_+^n$ is chosen at stage 1 of the Kreps–Scheinkman game then the following second stage subgame Nash equilibrium always induces expected payoffs equal to the Cournot payoffs following q. Formally, denote by $(\mu_1, \ldots, \mu_n)$ a vector of mixed strategies for the stage 2 subgame (i.e. an n-vector of probability measures on R_{++}), and let μ denote the product measure $\mu = \prod \mu_i$ on R_{++}^n.

Definition. The Kreps–Scheinkman model has the exact Cournot reduced form if for all $q \in R_+^n$ and for all mixed strategy Nash equilibria $(\mu_1, ..., \mu_n)$ of the stage 2 subgame following q,

$$\int_{R_{++}^n} R_i(q,p) \cdot d\mu(p) - c_i(q_i) = \pi_i^c(q), \quad i = 1, \ldots, n$$

In this section we first show that the Kreps–Scheinkman model has the exact Cournot reduced form if demand is uniformly elastic, and then move on to show that such a demand specification is consistent with existence and uniqueness of pure strategy Cournot-Nash equilibrium. Finally, we show that a weaker equivalence of the Kreps–Scheinkman and Cournot models prevails if either the range of elastic demand or costs are sufficiently large.

It is obvious that proportional rationing will ensure that firms announcing the market-clearing price $p_i = F(Q), i = 1, \ldots, n$ will be a Nash equilibrium of the stage 2 Kreps–Scheinkman subgame, if Q is at an elastic point of the demand curve. No firm will lower price, since they can sell all available output at $F(Q)$. And raising price will capture a fraction of market demand, and hence the same fraction of market revenue, which falls as price is raised because of the elasticity; so no firm will raise price either. It is not so obvious that this is the only possible stage 2 equilibrium, at least in terms of payoffs, a result which has been proved (implicitly) by Allen and Hellwig (1986) in their detailed analysis of the Bertrand-Edgeworth game, a game which is exactly the second stage of the Kreps–Scheinkman game. We now show that this result survives our more general rationing scheme.

Theorem 1. Suppose that assumptions 1 (or 2) and 3 hold and that q with $\Sigma q_k = Q$ is given at stage 1 of the Kreps–Scheinkman game. If demand is elastic at Q (i.e. $Q \leq D(a)$) then expected revenue in any Nash equilibrium of the stage 2 subgame following Q is $\pi_i^c(q), i = 1, \ldots, n$

If $Q = 0$, this result is immediate from the definitions. If $Q > 0$ it is proved via the following lemmas.

Lemma 1. Suppose that assumptions 1 (or 2) and 3 hold, and that q with $\Sigma q_k = Q > 0$ is given at stage 1. Then in the following stage-2 game:
(a) the pure strategy $p_i = F(Q)$ guarantees firm i a revenue of $q_i F(Q)$
(b) any price $p_i < F(Q)$ is strictly dominated for firm i, if $q_i > 0$.

Lemma 2. Suppose that assumptions 1 (or 2) and 3 hold and that q with $\Sigma_{qk} = Q > 0$ is given at stage 1. Suppose that demand is elastic at Q (i.e. $Q \leq D(a)$. Then the pure strategies $p_i = F(Q), i = 1, \ldots, n$ are a Nash equilibrium of the stage 2 subgame following Q.

Remark. The essence of lemma 2 is that a Nash deviation in which a firm raises price from the suggested equilibrium cannot be beneficial since it earns the firm at most the same share of market revenue at the higher price which is lower because of the elasticity. The sunk cost assumption is critical here. Were output decisions made after stage one capacity and after stage two prices and demand have been realised (as in the Kreps and Scheinkman (1983, p. 337) suggestion) lemma 2 would cease to hold since the deviating firm would truncate production to within capacity and gain cost-saving benefits which can compensate for the revenue loss (and would compensate e.g. if demand was unit elastic and rationing was proportional).

Lemma 3. Under the suppositions of lemma 2, expected revenues in any Nash equilibrium of the stage 2 subgame following Q are $q_i F(Q), i = 1, \ldots, n$.

Of course, if demand is uniformly elastic ($a = 0$ so $D(a) = \infty$), then the conclusion of Theorem 1 applies following any first stage quantities. Thus:

Theorem 2. Suppose that assumptions 1 (or 2) and 3 hold and that demand is uniformly elastic (i.e. $a = 0$). Then the Kreps–Scheinkman model has the exact Cournot reduced form.

For theorem 2 to provide a compelling defence of the Cournot specification one needs to know that its assumptions are consistent with existence of Nash equilibrium in the Cournot model (= perfect equilibrium in Kreps–Scheinkman under Theorem 2's conditions). In fact relatively innocuous additional "convexity" and "boundary" assumptions are sufficient to guarantee existence of a pure strategy Cournot-Nash equilibrium, which is also unique if costs are symmetric. This is reported in Theorem 3, whose existence argument is non-standard because of the payoff discontinuity allowed by our specification at $Q = 0$. The discontinuity entails a failure of upper semi-continuity so the results of Dasgupta and Maskin (1986) are not directly applicable. The argument used perturbs the Cournot game by insisting that the lowest cost firm produce at least $\epsilon > 0$. The perturbed game then has an equilibrium (by standard arguments). Letting $\epsilon \to 0$, it is possible to show that the ϵ constraint becomes non-binding in an equilibrium of the perturbed game (there must be at least 2 firms for this to follow), which is then an equilibrium of the original game. The extra assumptions used are as follows.

Assumption 4. $\lim_{Q \to \infty} \psi'(Q) = 0$

Assumption 5. $\psi : R_{++} \to R_{++}$ is concave on $(0, b)$

Assumption 6. For each $i = 1, \ldots, n$, c_i is a convex, strictly increasing function.

In the constant elasticity example introduced earlier $\psi(Q) = Q^{1-\frac{1}{\varepsilon}}$, $\varepsilon \geq 1$, and assumptions 4 and 5 are certainly satisfied.

Theorem 3. Suppose that assumptions 1 (or 2) and 3–6 hold, and that demand is uniformly elastic (i.e. $b = \infty$). Then;

(a) there exists at least one pure strategy Cournot-Nash equilibrium, and
(b) there is a unique, pure strategy Cournot-Nash equilibrium if costs are symmetric (i.e. $c_1 = c_2 = \ldots = c_n$).

It is however clear that theorem 2 in its exact form does not survive to situations where demand becomes inelastic eventually ($a > 0$). Indeed, for the proportional rationing case, the results of Allen and Hellwig (1986) imply that for any $Q > D(a)$ the stage 2 equilibria involves non-degenerate mixed strategies (with supports in $[F(Q), a]$), precluding Theorem 1 at inelastic points of the demand curve and thus precluding the strong exact Cournot reduced form result of theorem 2. A slightly weaker result than theorem 2 (but still strong enough that the Cournot-Nash equilibria coincide with the perfect equilibria of the Kreps–Scheinkman model) is available however, either when the range over which demand is elastic is sufficiently large, or when costs are sufficiently large.

Suppose it can be shown that for firm $i = 1, \ldots, n$ any strategy where $q_i > \hat{q}_i$ is strictly dominated in both the Cournot and Kreps–Scheinkman models; truncate admissible quantities to $[0, \hat{q}_i]$ to get the *truncated Cournot and Kreps-Scheinkman models*.

Definition. If for firm $i = 1, \ldots, n$ any strategy where $q_i > \hat{q}_i$ is strictly dominated in both the Cournot and Kreps–Scheinkman models then the truncated Kreps–Scheinkman model has the exact truncated Cournot reduced form if, for all $q \in \times_{i=1}^{n} [0, \hat{q}_i]$ and for all mixed strategy Nash equilibria $(\mu_1, \ldots, \mu_n)$ of the stage 2 game following q,

$$\int_{R^n_{++}} R_i(q,p) \cdot d\mu(p) - c_i(q_i) = \pi_i^c(q), \quad i = 1, \ldots, n$$

Since removal of strictly dominated strategies will affect neither the perfect equilibria of the Kreps-Scheinkman model nor the Cournot-Nash equilibria, these two sets of equilibria will coincide if the truncated Kreps-Scheinkman model has the exact truncated Cournot reduced form. To establish such a "truncated" result we adopt parameterizations of demand and costs.

Assumption 7. The market revenue function in terms of quantity, $\psi : R_{++} \to R_{++}$ is given by $\psi(Q) = \hat{\psi}(Q, \alpha)$ where $\alpha > 0$ is a parameter and $\hat{\psi}(Q)$ and its inverse demand function $\hat{\psi}(Q)/Q$ satisfy assumptions for 1 (or 2), 4 and 5 with $a = F(b) > 0$.

The effect of changing α is to leave the maximum value of ψ unchanged whilst changing the range of output over which demand is elastic (ψ is non-decreasing) to $(0, \alpha D(a))$; thus "large" α correspond to a more wide-ranging elasticity of demand.

Assumption 8. For $i = 1, \ldots, n$ the cost function $c_i : R_+ \to R_+$ is given by $c_i(q_i) = \beta \hat{c}_i(q_i)$ where $\hat{c}_i, i = 1, \ldots, n$ satisfy assumption 6, and where $\beta > 0$ is a parameter.

Clearly large β means large costs. The following is proved in the appendix.

Theorem 4. Suppose assumptions 3, 7 and 8 are satisfied. Then the truncated Kreps–Scheinkman model has the exact truncated Cournot reduced form if either α or β is sufficiently large; in either case, there exists a pure strategy Cournot-Nash equilibrium which is unique if costs are symmetric.

Remark. On the other hand fixing α and letting β become small will eventually force the Cournot-Nash equilibrium to an inelastic point on the demand curve. Then it is straightforward to check that the Cournot-Nash equilibrium ceases to be a perfect equilibrium of the Kreps-Scheinkman model under proportional rationing – any firm would wish to deviate in the second stage subgame to a price higher than $F(Q)$ since such deviations provide a constant share of an increased market revenue with inelastic demand and proportional rationing. And by the time the limit where $\beta = 0$ is reached, the Davidson-Deneckere result emerges – the Cournot-Nash equilibrium ceases to be a perfect equilibrium under any rationing scheme apart from the efficient one.

4 Concluding remarks

A well-known lacuna of the classic Cournot model is its failure to provide a rationalisation for the pricing decisions which accompany the Nash quantity choices. We have shown here (inter alia) that appending a second stage pricing game to a first-stage quantity choice à la Kreps–Scheinkman (1983) rescues the Cournot model quite generally if all costs are sunk at the first stage and if demand is uniformly elastic. In particular the conclusion is robust against choice of rationing rule. We suggest that the Cournot story is much more compelling in the context of markets characterised by such sunk cost and elastic demand features.

Appendix

Proof of Lemma 1. (a) This is trivial if $q_i = 0$. Suppose $q_i > 0$. When firms name the vector p with $p_i = F(Q)$,

$$\Delta_i(q,p) \geq \Delta_{iE}(q,p) = \max\left\{0, \left[D(p_i) - \sum_{p_k<p_i} q_k\right] \cdot \frac{q_i}{\sum_{p_k=p_i} q_k}\right\}$$

$$= \max\left\{0, \sum_{p_k \geq p_i} q_k \cdot \frac{q_i}{\sum_{p_k=p_i} q_k}\right\} \geq q_i$$

Hence $R_i(q,p) = p_i q_i = q_i F(Q)$ in any such realisation.
(b) If $p_i < F(Q), R_i(q,p) \leq p_i q_i < q_i F(Q)$ if $q_i > 0$, and (a) ensures that such prices are strictly dominated in the stage 2 subgame.

Proof of Lemma 2. First note that if $q_i = 0$, all stage 2 strategies produce an expected revenue of zero for firm i; thus $p_i = F(Q)$ is always a best response at stage 2 if $q_i = 0$. Suppose now that $p_i = F(Q)$ for all $i = 1, \ldots, n$ and consider a firm where $q_i > 0$; without loss of generality $i = 1$. Suppose firm 1 deviates to a higher price, $\hat{p}_1$ say Let $\hat{p}$ denote the price vector with first component $\hat{p}_1$, all others remaining at $F(Q)$. Then, from assumption 3, $\hat{p}_1 \Delta_1(q,\hat{p}) \leq \hat{p}_1 \Delta_{1P}(q,\hat{p})$. Also $p_1 \Delta_{1E}(q,p) = p_1 \Delta_{1P}(q,p) - p_1 \Delta_1(q,p)$ from assumption 3 and the functional forms of Δ_{1E}, Δ_{1P}. Now,

$$\hat{p}_1 \Delta_{1P}(q,\hat{p}_1) = \hat{p}_1 \left[1 - \sum_{k=2}^{n} \frac{q_k}{D(F(Q))}\right] D(\hat{p}_1)$$

which is non-increasing in $\hat{p}_1$ since demand is elastic at $\hat{p}_1 > p_1$. Also,

$$\lim_{\hat{p}_1 \to p_1^+} \hat{p}_1 \Delta_{1P}(q,\hat{p}) = \left[1 - \sum_{k=2}^{n} (q_k/Q)\right] \cdot F(Q) \cdot Q$$

and $p_1 \Delta_{1P}(q,p) = F(Q) \cdot Q \cdot (q_{P_1}/Q) = \lim_{\hat{p}_1 \to p_1^+} \hat{p}_1 \Delta_{1P}(q,\hat{p})$. Thus $p_1 \Delta_{1P}(q,p) = p_1 \Delta_1(q,p) \geq \hat{p}_1 \Delta_{1P}(q,\hat{p}) \geq \hat{p}_1 \Delta_1(q,\hat{p})$, for $\hat{p}_1 \geq p_1$. So raising p_1 from $F(Q)$ to $\hat{p}_1$ cannot be beneficial for firm 1, and since lowering p_1 from $F(Q)$ is clearly not beneficial, $p_i = F(Q), i = 1, \dots, n$ must be a Nash equilibrium of the stage 2 subgame.

Proof of Lemma 3. For a firm with $q_i = 0$, all stage 2 strategies produce an expected revenue of zero $(= q_i F(Q))$. Moreover since prices named by such firms have no effect on other firm's revenues (assumption 3(ii)) there is no loss of generality in assuming $p_i = F(Q)$ if $q_i = 0$. From lemma 1 we can be sure that the support of any strategy of any firm in a stage 2 Nash equilibrium is contained in $[F(Q), \infty]$. Consider any p where $p_i \geq F(Q), i = 1, \dots, n$, and use the following notation; let $p^1 < p^2 < \dots < p^s$ denote the $s(\leq n)$ different prices in p, and suppose p^j is the price of firms in the set N^j (with $\bigcup_{j=1}^{s} N^j = \{1, \dots, n\}$) and $N^i \bigcap N^j = \phi, i \neq j$) where $\sum_{i \in N^j} q_i = Q^j$, say. Now, for $i = 1, \dots, n; R_i(q,p) = p_i \min(q_i, \Delta_i(q,p)) \leq p_i \min(q_i, \Delta_{iP}(q,p))$

$$= p_i \min\left\{ q_i, \max\left[0, 1 - \sum_{p_k < p_i} \frac{q_k}{D(p_k)}\right] \cdot D(p_i) \cdot \frac{q_i}{\sum_{p_k = p_i} q_k} \right\}$$

Hence, for $i \in N^j, j = 1, \dots, s$;

$$R_i(q,p) \leq p^j \min\left\{ q_i, \max\left[0, 1 - \sum_{h<j} \frac{Q^h}{D(p^h)}\right\} \cdot D(p^j) \cdot \frac{q_i}{Q^j} \right\}$$

We now establish that aggregate revenue, $\sum_{i=1}^{n} R_i(q,p)$ cannot exceed $QF(Q)$. Suppose first that $Q^1/D(p^1) > 1$. Then for $i \in N^1, R_i(q,p) = p^1 D(p^1) \cdot (q^i/Q^1)$ and $R_i(q,p) = 0$ for $i \notin N^1$; so $\sum R_i(q,p) = p^1 D(p^1) \leq Q(F(Q)$, since demand is elastic at Q and since $p^1 \geq F(Q)$. Suppose next that there is an integer $r, 0 < r < s$ such that

$$\sum_{h=1}^{j} Q^h / D(p^h) \begin{cases} \leq 1 \text{ for } j = 1, \dots, r \\ > 1 \text{ for } j = r+1, \dots, s \end{cases}$$

Then, for $i \in N_j, j = 1, \dots, r, R_i(q,p) \leq p^j q_i$, noting that

$$\left[1 - \sum_{h=1}^{j-1} Q^h / D(p^h)\right] \cdot D(p^j)/Q^j \geq 1 \text{ if and only if}$$

$$1 - \sum_{h=1}^{j} Q^h / D(p^h) \geq 0. \text{ For } i \in N_{r+1};$$

$$R_i(q,p) \leq p^{r+1} \min\left\{ q_i, \left[1 - \sum_{h=1}^{r} \frac{Q^h}{D(p^h)}\right] \cdot \frac{q_i}{Q^{r+1}} \cdot D(p^{r+1}) \right\}$$

$$= p^{r+1} \left[1 - \sum_{h=1}^{r} \frac{Q^h}{D(p^h)}\right] \cdot \frac{q_i}{Q^{r+1}} \cdot D(p^{r+1})$$

since $1 - \sum_{h=1}^{r+1} Q^h/D(p^h) < 0$. And for $i \in N^j, j = r+2, ..., s,\ R_i(q,p) = 0$ since $1 - \sum_{h=1}^{j-1} Q^h/D(p^h) < 0$ when $j - 1 \geq r + 1$.
Summing, we conclude that;

$$\begin{aligned}\sum_{i=1}^{n} R_i(q,p) &\leq p^1Q^1 + \ldots + p^rQ^r + p^{r+1}D(p^{r+1})\left[1 - \sum_{h=1}^{r}\frac{Q^h}{D(p^h)}\right] \\ &= p^1D(p^1)\frac{Q^1}{D(p^1)} + \ldots + p^rD(p^r)\frac{Q^r}{D(p^r)} \\ &\quad + p^{r+1}D(p^{r+1})\left[1 - \sum_{h=1}^{r}\frac{Q^h}{D(p^h)}\right] \\ &\leq QF(Q)\end{aligned}$$

as it is a convex combination of $p^iD(p^i), i = 1, \ldots, r+1$, each of which cannot exceed $QF(Q)$ since demand is elastic at Q and $p_i \geq F(Q)$. Suppose finally that $\sum_{h=1}^{j} Q^h/D(p^h) \leq 1$ for all $j = 1, \ldots, s$. Since $D' < 0$ this can only occur if $p^h = F(Q), h = 1, \ldots, s$ and aggregate revenue is $Q \cdot F(Q)$. Thus, in all cases;

$$\sum_{i=1}^{n} R_i(q,p) \leq QF(Q)$$

Since aggregate revenue cannot exceed $QF(Q)$ at any price realisable in a stage 2 Nash equilibrium, it follows that aggregate expected revenue cannot exceed $QF(Q)$ in any stage 2 Nash equilibrium. From lemma 2 we know there is an equilibrium in which firm i receives revenue $q_iF(Q), i = 1, \ldots, n$. If there was an equilibrium in which expected revenue differs from $q_iF(Q)$ for some i, then for some firm k expected revenue must be strictly less than $q_kF(Q)$. But this contradicts lemma 1, and thus completes the proof of lemma 3.

Proof of Theorem 3(a). The best response problem of players i in the Cournot model is:

$$\max_{q_i} r_i(q) - c_i(q_i) \quad \text{s.t.} \quad q_i \geq 0 \qquad (BR_i)$$

where $r_i : R^n_+ \to R_+$ is defined by $r_i(q) = q_iF(Q)$. We can put an upper bound on the effective feasible set for (BR_i) as follows. Since c_i is convex and strictly increasing, and from assumption 4, there is a $\bar{\bar{q}}_i > 0$ such that $\psi'(q_i) - c'(q_i) < -\epsilon$ for all $q_i > \bar{\bar{q}}_i$ and for some $\epsilon > 0$. It follows that there exists $\bar{q}_i > \bar{\bar{q}}_i$, where $\psi(q_i) - c_i(q_i) < 0$ for all $q_i > \bar{q}_i$. Moreover $r_i(q) = q_iF(Q) \leq \psi(q_i) = q_iF(q_i)$ for all $q_i \geq 0$ since F is decreasing: hence $r(q_i) - c_i(q_i) < 0$ for all $q_i > \bar{q}_i$ and any choice of $q_i > \bar{q}_i$ is strictly dominated by $q_i = 0$, for instance. Thus we may take the feasible set for ($\mathrm{BR_i}$) as $[0, \bar{q}_i]$; we can now be sure that (BR_i) has a solution if $\sum_{j \neq i} q_j > 0$ since r_i is then continuous in $q_i \in [0, \bar{q}_i]$. To evade the discontinuity in r_i at $q_i = \sum_{j \neq i} q_j = 0$, suppose (without loss of generality) that $c_i'(0)$ reaches a minimum over i for $i = 1$ (i.e. $c_1'(0) \leq c_i'(0), i = 1, \ldots, n$), and consider the perturbed game where the only change is that q_1 is restricted to be at least ε, some small $\varepsilon > 0$ where $\varepsilon < \bar{q}_1$. Thus consider the perturbed problem for player 1:

$$\max_{q_1} r_1(q) - c_1(q_1) \quad \text{s.t.} \quad \varepsilon \leq q_1 \leq \bar{q}_1 \qquad (PBR_1)$$

To check for concavity of $r_i(q)$ note first that whenever $Q > 0$;

$$r_i''(q) = \frac{\partial^2 r_i}{\partial q_i^2} = 2F'(Q) + q_i F''(Q) \leq 0 \qquad \text{if} \quad F''(Q) \leq 0$$

Also by assumption 5, $\psi''(Q) = 2F'(Q) + QF''(Q) \leq 0$. Hence $r_i''(q) - \psi''(Q) = (q_i - Q)F''(Q) \leq 0$ if $F''(Q) \geq 0$, and so $r_i''(q) \leq 0$ also when $F''(Q) \geq 0$. Thus r_i is concave in $q_i = 1, ..., n$. The restriction to $q_i \geq \varepsilon > 0$ ensures that r_i is continuous (in fact C^2) in $q, i = 1, ..., n$. By standard arguments, the perturbed game then has at least one Nash equilibrium.

Suppose that the perturbed game has a Nash equilibrium q in which $q_1 = \varepsilon$ and $\Sigma q_k = Q$. From the first order conditions for solutions to best response problems we have:

$$F(Q) + \varepsilon F'(Q) - c_1'(\varepsilon) \leq 0 \tag{1}$$

and for $i = 2, ...n$ $\quad F(Q) + q_i F'(Q) - c_i'(q_i) \leq 0$ with equality if $q_1 > 0$ (2)

$$\text{Adding}: (n-1) \quad F(Q) + [F(Q) + QF'(Q)] - \sum_{i=1}^{n} c_i'(q_i) \leq 0 \tag{3}$$

The square bracket on the left hand side of (3) is non-negative because of uniform elasticity. Thus, assuming $n > 1$, (3) provides a finite upper bound for $F(Q)$ and a positive lower bound for $Q\left(\text{e.g. } Q \geq F^{-1}\left[\sum_{i=1}^{n} c_i'(\bar{q}_i)/(n-1)\right]\right)$ in such a Nash equilibrium of the perturbed game. Consider now an infinite decreasing sequence of positive ε values, say $\varepsilon_v, v = 1, 2, \ldots,$, where $\varepsilon_v \to 0$ as $v \to \infty$, and consider the corresponding sequence of perturbed games. Suppose $q = q_v$ and $Q = Q_v$ are equilibrium values at ε_v with $q_{1v} = \varepsilon_v$; suppose (we can take a converging subsequence if necessary) $Q_v \to Q^*$ and $q_v \to q^*$ as $v \to \infty$. Since Q_v is bounded away from 0, $Q^* > 0$ and (1) gives

$$F(Q)_v + \varepsilon_v F'(Q_v) - c_1'(\varepsilon_v) \leq 0$$

Hence, in the limit, $F(Q^*) \leq c_1'(0)$. Notice then that $F(Q^*) \leq c_i'(0) \leq c_i'(q_i^*)$ for any i.

Since $Q^* > 0$, $q_i^* > 0$ for some $i = 2, \ldots, n$. For such i, from (2);

$$F(Q^*) + q_i^* F'(Q^*) - c_i'(q_i^*) = 0$$

And this contradicts $F(Q^*) \leq c_i'(q^*)$. So for some ε_ν it must be that the Nash equilibria of the perturbed game have $q_{1\nu} > \varepsilon_\nu$. Since the objective in (PBR_1) is concave in $q_1 \in [0, \bar{q}_1]$ it follows that $q_{1\nu}$ also solves (BR_1) and hence that q_ν is a Nash equilibrium of the original, unperturbed game.

Proof of Theorem 3(b). Let $c = c_1 = \ldots = c_n$ be the cost function. Notice first that we cannot have $Q = 0$ in a pure strategy equilibrium, since, for instance $[\psi(Q) - c(Q)]/Q \to +\infty$ as $Q \to 0$, ensuring that positive profits can be made. Secondly each q_i must be strictly positive. Otherwise the first-order conditions for $q_i = 0$ (see (2) in the proof of Theorem 3(a)), $F(Q) \leq c'(0)$ and the first-order conditions for $q_i > 0$ cannot be satisfied. Hence, for each $i, F(Q) + q_i F'(Q) = c'(q_i)$, which can have only one solution in q_i for any given Q. Thus equilibrium must be symmetric, $q_i = Q/n, i = 1, ..., n$ and

$$F'(Q) \cdot Q/n = c'(Q/n) \tag{4}$$

The slope of the left hand side of (4) is $\frac{1}{n}\{(n-1)F'(Q) + [2F'(Q) + QF''(Q)]\}$. The square bracket here is non-positive from assumption 5 and so this slope is everywhere strictly negative when $n > 1$ since $F'(Q) < 0$. The slope of the right hand of (4) is non-negative from assumption 6 and so (4) has at most one solution, which ensures uniqueness.

Proof of Theorem 4. We show the result for sufficiently large α. The argument for β is similar and left to the reader. Suppose (without loss of generality) that $\beta = 1$. Let $M = \max_{Q \geq 0} \psi(Q) = \max_{Q \geq 0} \hat{\psi}(Q/\alpha)$ – note that $M > 0$ and does not vary with α. For each i define $\hat{q}_i$ uniquely (since $\hat{c}_i$ satisfies assumption 6) by $\hat{c}_i(\hat{q}_i) = M$. Then for any i and q, if $q_i > \hat{q}_i$,

$$\pi_i^c(q) = q_i F(Q) - \hat{c}_i(q_i) < q_i F(Q) - M \leq 0,$$

since $q_i F(Q) \leq Q\, F(Q) \leq M$ for all Q. Also,

$$\pi_i^{KS}(q, p) = R_i(q, p) - \hat{c}_i(q_i) < R_i(q, p) - M \leq 0$$

since $R_i(q, p) \leq M$ for all $q \in R_+^n, p \in R_{++}^n$ in the Kreps–Scheinkman model. It follows that strategies in $(\hat{q}_i, \infty)$ are strictly dominated in both the Cournot and Kreps–Scheinkman models. Now define α^* by $\alpha^* D(a) = \sum_{i=1}^n \hat{q}_i$. If $q_i \leq \hat{q}_i, i = 1, \ldots, n$ and $\alpha \geq \alpha^*$ then $Q = \sum_{i=1}^n q_i \leq \alpha D(a)$ and Q corresponds to a point on the demand curve where demand is elastic. Theorem 1 then proves the exact reduced form result.

Finally existence follows as in Theorem 3(a), with $\hat{q}_i$ replacing $\bar{q}_i$ since demand is elastic at any feasible $Q = \sum_{i=1}^n q_i$, and uniqueness follows as in Theorem 3(b) if costs are symmetric.

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