The Stability of the Cournot Oligopoly Solution
Author(s): F. H. Hahn
Source: *The Review of Economic Studies*, Vol. 29, No. 4 (Oct., 1962), pp. 329-331
Published by: Oxford University Press
Stable URL: https://www.jstor.org/stable/2296310
Accessed: 22-12-2019 11:28 UTC

The Stability of the Cournot Oligopoly Solution

Recent months have seen a number of papers on this topic. [1. 2. 3.] They all involve more or less lengthy and complex discussions of roots of matrices. They also make the strong assumption that all cost functions are identical for all producers and that marginal costs and market price are both linear in output. This note shows how these assumptions can be dropped and it also derives all results by extremely simple methods involving neither matrices nor determinants. The note is offered not in the belief that the problem as such is of the slightest direct importance but that its study may suggest fruitful extensions in due course. If that is so then perhaps something will be gained by a short note making fewer demands on the reader's patience and not making assumptions stronger than the ones already imposed by the nature of the problem.

(1) Let X_i = output of the ith competitor and write $X = \Sigma X_i$ (*not* a vector). Also define $X^k = \sum_{i \neq k} X_i$. Evidently we are supposing all outputs to be homogeneous and to sell at the same price P. The market demand function is written as

$$P = d(X),\ d(0) > 0,\ d' < 0 \tag{1}$$

Each producer i has total cost C_i which depends on his own output only, i.e.

$$C_i = C_i(X_i) \tag{2}$$

All cost functions and the demand function are supposed to be continuously differentiable for as often as is necessary. The profit maximising output X_i^* of the ith Cournot producer is found by solving

$$d(X) + X_i d'(X) - C_i'(X_i) = 0 \tag{3}$$

where the output of all other producers is taken as given. We suppose $d(0) - C'(0) < 0$. We also assume that there exists a unique solution[1] $X_i^* \geqslant 0$ to (3). Second order conditions for the individual producer are

$$2d' + X_i^* d'' - C_i'' < 0 \tag{4}$$

We are asked to suppose that actual output may diverge from profit maximising output because of adjustment delays. In particular we are given the following adjustment system:

$$\dot{X}_i = K_i(X_i^* - X_i) \qquad i = 1 \ldots n,\ K_i > 0 \tag{5}$$

putting $x_i \equiv (X_i^* - X_i)$ it will be tidier to write

[1] The assumptions here are more strict than are necessary. The producer has always the choice of not producing at all. The equilibrium condition is

$$d(X) + X_i d'(X) - C_i'(X_i) \leqslant 0$$

with the inequality only if $X_i^* = 0$. In that case condition (4) only holds if the above first order condition holds with equality. But as long as there is a strict inequality in the first order condition we set $\dot{X}_i^{*i} = 0$ in (8) and the analysis proceeds just as before.

$$\dot{X}_i = K_i x_i \tag{5'}$$

The problem is to discover whether (5′) leads to an adjustment towards an oligopoly equilibrium.

(2) The case so far investigated is peculiarly simple. It is supposed that the demand function is linear in X, i.e. $d'' = 0$, that all producers face the same marginal cost functions and that C_i is linear in X_i. Let us put this formally.

*Assumption A.*1

$$d'' = 0, \; C_i' = C'' \text{ a constant all } i.$$

Now consider

$$2V = \sum_i K_i (x_i)^2 \tag{6}$$

Evidently $V \geqslant 0$ and $V = 0$ if and only if $x_i = 0$ all i. We wish to investigate the sign of $\dot{V}$. If the latter is negative if and only if $V > 0$ then the system will evidently approach an oligopoly equilibrium arbitrarily closely. Using (5′) we have

$$\dot{V} = \Sigma K_i x_i (\dot{X}_i^* - \dot{X}_i) = -\Sigma (K_i x_i)^2 + \Sigma K_i x_i \dot{X}_i^* \tag{7}$$

Differentiate (3) totally with respect to t, use A.1 and solve for $\dot{X}_i^*$, i.e.

$$\dot{X}_i^* = -q\dot{X}^i \tag{8}$$

where $q = \dfrac{d'}{2d' - C''} < 0$ by (1) and (4). Using (5′), (8) becomes

$$\dot{X}_i^* = -q \sum_{j \neq i} K_j x_j \tag{9}$$

Substituting in (7) we have

$$\dot{V} = -[\Sigma (K_i x_i)^2 + q \sum_i K_i x_i \sum_{j \neq i} K_j x_j] \tag{10}$$

Now if the first term in square brackets is zero then so is the second. If the second term is otherwise always positive then the sign of the square bracket is always the same as the sign of the first term in that bracket. Hence we need only worry about the possibility of the term multiplying q being negative. Suppose it is. Assume $q < 1$. Then we have for $V \neq 0$

$$-\dot{V} > [\sum_i (K_i x_i)^2 + \sum_i K_i x_i \sum_{j \neq i} K_j x_j] = (\sum_j K_j x_j)^2 \geqslant 0 \tag{11}$$

Hence once again if $\sum_i (K_i x_i)^2 \neq 0$, the sign of $-\dot{V}$ is positive, and $\dot{V} = 0$ if and only if $\sum_i (K_i x_i)^2 = 0$. Hence a sufficient condition for stability is $q < 1$ or

$$d' < C'' \tag{12}$$

This will always be so if $C'' \geqslant 0$. This is Fisher's condition [2] (3.9)

(4) We now drop A.1. Proceeding as before we rewrite (8) as

$$\dot{X}_i^* = -q_i \dot{X}^i \tag{8'}$$

where

$$q_i = \frac{d' + X_i^* d''}{(2d' + X_i^* d'' - C_i'')} \tag{13}$$

We note that q_i is a function of X. Its denominator is negative by (4). We now introduce

Assumption A.2

$$d' + X_i^* d'' < 0 \text{ for all possible } X.$$

A.2 states that at all possible outputs the marginal revenue of any one producer, at a given output of his, is a diminishing function of total industry output. This seems reasonable.

We now proceed exactly as before to find

$$\dot{V} = -\sum_i [(K_i x_i)^2 + q_i K_i x_i \sum_{j \neq i} K_j x_j] \tag{10'}$$

From our point of view the worst that could happen is that *every* term involving q_i is negative. Suppose that to be so. Then if max $q_i < 1$ all X we may again form the inequality (11) and once again $\dot{V} < 0$. Hence a general sufficient condition for stability is

$$d' < C_i'' \text{ all } i \text{ and } X.$$

Cambridge. F. H. HAHN

REFERENCES

[1] R. D. Theocharis: " On the Stability of the Cournot Solution of the Oligopoly Problem," *R.E.S.*, Vol. XXVII (2), 1959-60.

[2] F. M. Fisher, " The Stability of the Cournot Oligopoly Solution: The Effects of Speeds of Adjustment and Increasing Marginal Costs," *R.E.S.*, Vol. XXVIII (2), 1961.

[3] M. McManus and R. E. Quandt, " Comments on the Stability of the Cournot Oligopoly Model," *R.E.S.*, Vol. XXVII (2), 1961.