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Price Competition, Quality and Income Disparities

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A market is considered, the demand side of which consists of a large number of consumers with identical tastes but different income levels, and the supply side, of two firms selling at no cost products which are relatively close substitutes for each other. Consumers are assumed to make indivisible and mutually exclusive purchases. A full characterization of the demand structure and the non cooperative market solution is given, and the dependence of the latter on income distribution and quality parameters is analyzed.

1. A HEURISTIC INTRODUCTION

In this paper we consider a market the demand side of which consists of a large number of consumers with identical tastes but different income levels, and the supply side of two firms selling at no cost products which are relatively close substitutes for each other. Consumers are assumed to make indivisible and mutually exclusive purchases. Accordingly, consumers choice operates on a finite number of "price-quality" alternatives made available to them by the firms competing in the industry. Doing so, we try to capture an important fact of real life: in many economic decisions, it seems that the quality component of the choice bears as much on the outcome of the choice as its quantity component. Sometimes, it even happens that only the quality component plays a role: this is necessarily the case if the choice of a consumer concerns indivisible products which, by their very nature, are either bought in a single unit of a single brand, or not bought at all. So are cars, TV's, washing machines, stereo chains, pianos, a.s.o.

To illustrate the issues analyzed in this paper, let us consider the apology of Mr. Smith who contemplates the opportunity of buying a new piano. The relevant question for him is not how many pianos to buy but rather whether he should buy a piano and, if yes, whether it should be a piano of brand A or a piano of brand B. Assume that Mr. Smith definitely ranks a piano of brand A higher than a piano of brand B and let p_A and p_B be the prices of,

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respectively, a piano A and B. Finally let R denote Mr. Smith's income. Under which conditions will Mr. Smith decide to buy, say, a piano A? First it is necessary that

$$U(0, R) \leqslant U(A, R - p_A), \tag{1.1}$$

where $U(0, R)$ denotes the utility of having an income R and no piano A, and $U(A, R - p_A)$ denotes the utility of having a piano A and an income R deflated of its price. Indeed if the converse of (1.1) would hold, then Mr. Smith would find preferable to keep unchanged his income R and play on his old piano, rather than paying as much as p_A and having a new piano A. Call the *A-reservation price* (resp. *B-reservation price*) of Mr. Smith the value π_A (resp. π_B) of p_A (resp. p_B) which makes Mr. Smith indifferent between the two issues. Clearly the value π_A (resp. π_B) must verify $U(0, R) = U(A, R - \pi_A)$ (resp. $U(0, R) = U(B, R - \pi_B)$.). Of course, according to Mr. Smith's preferences, $\pi_A > \pi_B$.

To go from individual to aggregate market behaviour, let us now imagine the simplest situation where all the pianists in the world would have an income identical to Mr. Smith and would completely agree with his preferences. What market solution will emerge from such a situation? Let p_B be any price quoted by the seller of piano B and assume that the seller of piano A quotes a price $\bar{p}_A$ such that, according to the common preferences of the pianists, $\bar{p}_A < \pi_A$ and $U(A, R - \bar{p}_A) > U(B, R - p_B)$. It is intuitively clear that, with such a pair of prices, all the pianists in the world will buy a piano A so that the whole pianistic industry will be under control of seller A. This simple reasoning shows that no room is left here for another piano seller in the industry, except if he would enter the market with a piano C of higher quality, which beats piano A in the common hierarchy.

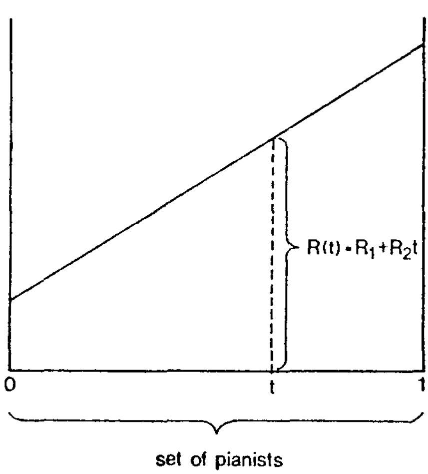

FIGURE 1

Of course, the reasoning does not go so simple if the preceding perfect symmetry is abandoned. In particular, since we are interested in this paper to examine the impact of income dispersion on product differentiation, let us abandon the assumption that all pianists have identical income. Let us however keep the assumption of identical preferences. To represent this situation, consider Figure 1 where the abscissa represents the set of pianists *ranked by order of increasing income* on the unit interval, and the ordinate represents their corresponding income levels so that $R(t)$ is the income of pianist t; we assume also

$$R(t) = R_1 + R_2 t \qquad (R_1 > 0, R_2 \geqslant 0)$$

(this amounts to specify a particular *uniform* distribution of income).

Furthermore, let us assume that the common preferences of our pianists are defined by the utility function

$$U(0, R(t)) = U_0 \cdot R(t),$$
$$U(A, R(t)) = U_A \cdot R(t),$$
$$U(B, R(t)) = U_B \cdot R(t),$$

where U_0, U_A and U_B are positive scalars verifying $U_A > U_B > U_0$; of course these inequalities reflect the hierarchy according to which a piano A is preferred to a piano B which in turn is preferred to nothing. Even with identical preferences, pianists reservation prices are no longer identical since they will now vary in accordance with their income. More precisely, the A-reservation price $\pi_A(t)$ of pianist t obtains from the condition $U(0, R(t)) = U(A, R(t) - \pi_A(t))$, or $U_0 \cdot (R_1 + R_2 t) = U_A \cdot (R_1 + R_2 t - \pi_A(t))$, i.e.,

$$\pi_A(t) = \frac{U_A - U_0}{U_A} \cdot (R_1 + R_2 t).$$

Similarly, the B-reservation price $\pi_B(t)$ of pianist t obtains from the condition $U(0, R(t)) = U(B, R(t) - \pi_B(t))$, i.e.,

$$\pi_B(t) = \frac{U_B - U_0}{U_B} \cdot (R_1 + R_2 t).$$

Accordingly, both A- and B-reservation prices are linear functions of t.

Under the preceding assumption, it is of course no longer true that when facing prices p_A and p_B quoted by the piano sellers all the pianists will act "in unison" ! To find out how the market is split at prices p_A and p_B, consider first Figure 2 where the sloping lines represent the magnitudes $\pi_B(t)$ and $(U_A/U_B)\,\pi_A(t)$ and the horizontal lines represent the levels p_B and $(U_A/U_B)\,p_A$.

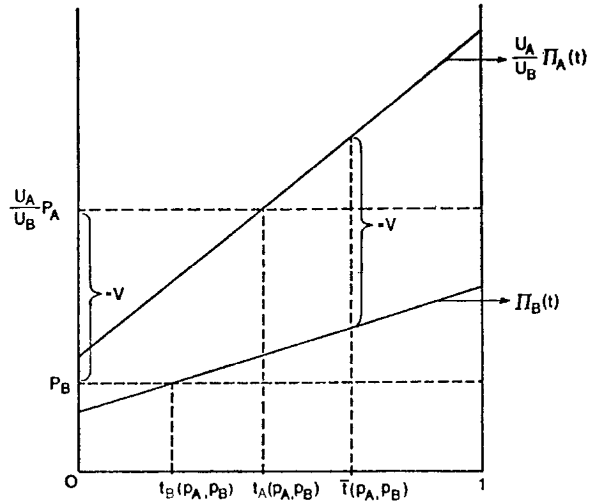

FIGURE 2

Clearly all pianists located at the left of the pianist $t_B(p_A, p_B)$ do not buy anything: for each of them, both their A- and B-reservation prices are smaller than the corresponding prices quoted by the piano sellers and they prefer to keep their income intact. As for the pianists located between $t_B(p_A, p_B)$ and $t_A(p_A, p_B)$, it is easy to see that they will buy a piano B, but not a piano A: their B-reservation price is larger than p_B, but their A-reservation price is still smaller than p_A. Consider now the set of pianists located at the right of $t_A(p_A, p_B)$: they all buy a piano, and they buy a piano A if, and only if, $U(A, R(t) - p_A) \geqslant U(B, R(t) - p_B)$. On the contrary, they buy a piano B. A simple reasoning shows that

$$U(A, R(t) - p_A) \geqslant U(B, R(t) - p_B) \Leftrightarrow U_A p_A - U_B p_B \leqslant U_A \pi_A(t) - U_B \pi_B(t).$$[1]

Consequently, returning to Figure 2, the "frontier" between those pianists who buy a piano B and those who buy a piano A is located at $\bar{t}(p_A, p_B)$ where the equality $V = (U_A/U_B) p_A - p_B = (U_A/U_B) \pi_A(\bar{t}(p_A, p_B)) - \pi_B(\bar{t}(p_A, p_B))$ is exactly verified.

Consider now Figure 3 where other values of p_A and p_B are represented. For these values, all the pianists are now willing to buy a piano since, for all of them, at least one of their reservation prices dominates the corresponding quoted price. In this situation, the whole market is served, and both sellers A

[1] Indeed: $U(A, R(t) - p_A) \geqslant U(B, R(t) - p_B) \Leftrightarrow U_A \cdot [(R(t) - \pi_A(t)) + (\pi_A(t) - p_A)] \geqslant U_B \cdot [(R(t) - \pi_B(t)) + (\pi_B(t) - p_B)] \Leftrightarrow U_A \cdot [\pi_A(t) - p_A] \geqslant U_B \cdot [\pi_B(t) - p_B]$, since $U_A \cdot (R(t) - \pi_A(t)) = U_B \cdot (R(t) - \pi_B(t)) = U_0 \cdot R(t)$.

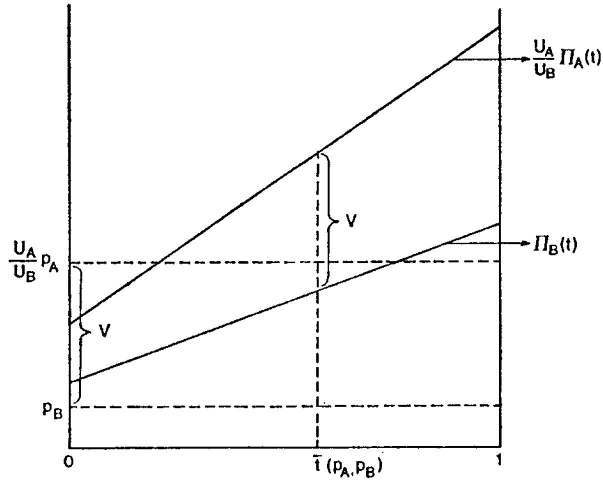

FIGURE 3

and B are in the market: seller B sells pianos to pianists in the interval $[0, \bar{t}(p_A, p_B)[$ and seller A to pianists in the interval $[\bar{t}(p_A, p_B), 1]$.

Finally consider Figure 4 where p_B is assumed to be equal to zero. All the pianists are again willing to buy a piano, but none of them buys piano B: even quoting $p_B = 0$, seller B cannot avoid the frontier to fall into 0, since even for the "poorest" pianist $t = 0$, we have $(U_A/U_B)\, p_A - p_B > (U_A/U_B)\, \pi_A(0) - \pi_B(0)$.

It follows from the preceding analysis that there are typically three price regions: a first region where both sellers A and B are in the market but with potential customers who are not served (Figure 2); a second region where again both sellers A and B are in the market and all customers are served

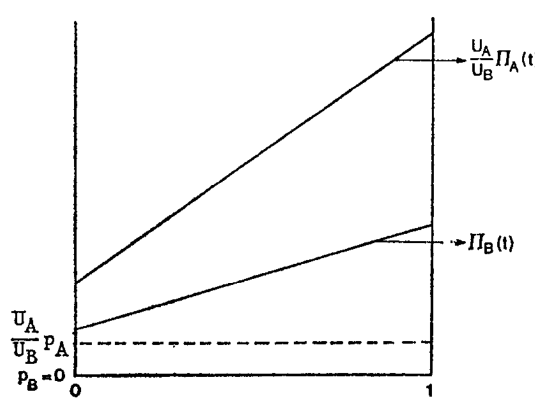

FIGURE 4

(Figure 3); and finally a third region where seller B is out of the market (Figure 4).

A first interesting issue which motivates the preceding analysis consists in identifying in which of these regions the two sellers will quote their prices "at equilibrium". To the extent that they play a noncooperative game with price strategies, an interesting concept of equilibrium to investigate is a Cournot price equilibrium where both sellers quote prices which are "best replies" to each other. The interest of this study arises from the fact the market structure can essentially differ according to the region in which such an equilibrium pair of prices would fall. For instance, if the Cournot equilibrium falls in the third region, it would mean that only pianos A can be sold to the customers, so that no room is left for the "standard" product piano B.

Another interesting issue, which is not completely distinct from the previous one, deals with the dependence of the price regions and corresponding equilibrium prices on the basic parameters of the game, namely, the "taste parameters" U_A and U_B, and the "income parameters", R_1 and R_2. It is clear indeed that both the demand functions depend on these parameters through the A- and B-reservation prices. Accordingly, the profit functions themselves depend on these parameters, so that equilibrium prices in turn are related to these values. The interest of studying this dependence arises from the fact that it is possible to conduct a comparative static analysis showing how equilibrium prices change when income distribution, or tastes, are modified. For instance, if the Cournot equilibrium moves from the second region to the third one when R_2 is decreased (with constant R_1), it would mean that a decrease in income dispersion leads to eliminate seller B from the market at the equilibrium. Or if U_A is close to U_B, it could mean that Cournot equilibrium prices would be close to each other and simultaneously near to zero: in that case we would have confirmation of Bertrand result for pure homogeneous products. In the next section, the preceding issues are formally analyzed in the framework of a model where the major features of the present heuristic introduction are maintained. In a short conclusion, we examine the possible extensions of the model and compare our paper with the existing literature.

2. A Formal Analysis

2.a. *The Model*

Let us first recall the basic ingredients of the preceding section. Consider a market with two duopolists each selling at no cost a product which is a more or less close substitute for the other, to a continuum of customers. Denote these products by A and B. By convention duopolist B sells the "standard" product at price p_B and duopolist A the "high quality" one at price p_A. Let $T = [0, 1]$ represent the set of customers. Assume that they are ranked in T by order of increasing income and that the income $R(t)$ is given by the linear relation:

$$R(t) = R_1 + R_2 t,\ t \in T,\ R_1 > 0,\ R_2 \geqslant 0.$$

All customers are assumed to have identical preferences defined by the utility function U, with $U(0, R(t)) = U_0 \cdot R(t)$, $U(A, R(t)) = U_A \cdot R(t)$ and $U(B, R(t)) = U_B \cdot R(t)$, $U_A > U_B > U_0 > 0$. We know from the preceding section that *A- and B-reservation prices of customer t* are defined, respectively, as

$$\pi_A(t) = \frac{U_A - U_0}{U_A}(R_1 + R_2 t) \tag{2.1}$$

and

$$\pi_B(t) = \frac{U_B - U_0}{U_B}(R_1 + R_2 t). \tag{2.2}$$

Before deriving the *demand functions* for each product, we have first to specify how, given the prices p_A and p_B quoted by the duopolists, the market T is partitioned between those who buy A (denote $M_A(p_A, p_B) =^{\mathrm{Def}} \{t \in T \mid t$ buys A at price $p_A\}$), those who buy B (denote $M_B(p_A, p_B) =^{\mathrm{Def}} \{t \in T \mid t$ buys B at price $p_B\}$), and those who buy neither A, nor B ($M_0(p_A, p_B)$).

LEMMA 1.

$$M_A(p_A, p_B) = \{t \in T \mid p_A \leqslant \pi_A(t)\} \cap \{t \in T \mid U_A p_A - U_B p_B \leqslant (U_A - U_B) R(t)\} \tag{2.3}$$

and

$$M_B(p_A, p_B) = \{t \in T \mid p_B \leqslant \pi_B(t)\} \cap \{t \in T \mid U_A p_A - U_B p_B > (U_A - U_B) R(t)\}. \tag{2.4}$$

The proof of this lemma easily follows from the argument developed in the introduction.

Our final task consists in deriving the demand function for each product as a function of prices p_A and p_B. Let μ be the Lebesgue measure defined on the set of measurable subsets of T and define $\mu_A(p_A, p_B) = \mu(M_A(p_A, p_B))$ and $\mu_B(p_A, p_B) = \mu(M_B(p_A, p_B))$. Let also S_A (resp. S_B) $= \{p_A \mid 0 \leqslant p_A \leqslant \pi_A(1)\}$ (resp. $\{p_B \mid 0 \leqslant p_B \leqslant \pi_B(1)\}$) be the *strategy set* of duopolist A (resp. duopolist B).[2] Finally define

[2] We have only to consider pairs of strategies in $S_A \times S_B$ since neither duopolist will choose his price strategy outside S_i. For instance, a value of p_A strictly larger than $\pi_A(1)$ implies that $\{t \in T \mid p_A \leqslant \pi_A(t)\} = \varnothing$, so that $\mu_A(p_A, p_B) = 0$, whatever the value of p_B. But then $p_A = \pi_A(1)$ would have done as well.

$$\mathscr{D}_1 = \{(p_A, p_B) | \mu_A(p_A, p_B) + \mu_B(p_A, p_B) < 1; \mu_A(p_A, p_B) > 0, \mu_B(p_A, p_B) > 0\}$$
$$\mathscr{D}_2 = \{(p_A, p_B) | \mu_A(p_A, p_B) + \mu_B(p_A, p_B) = 1; \mu_A(p_A, p_B) > 0, \mu_B(p_A, p_B) > 0\}$$
$$\mathscr{D}_3 = \{(p_A, p_B) | \mu_B(p_A, p_B) = 0; 0 \leqslant \mu_A(p_A, p_B) \leqslant 1\}.$$ [3]

LEMMA 2. *If* $(p_A, p_B) \in \mathscr{D}_1$, *then*

$$\begin{aligned} \mu_A(p_A, p_B) &= 1 - \frac{U_A p_A - U_B p_B}{(U_A - U_B) R_2} + \frac{R_1}{R_2}; \\ \mu_B(p_A, p_B) &= \frac{U_A p_A - U_B p_B}{(U_A - U_B) R_2} - \frac{U_B p_B}{(U_B - U_0) R_2}. \end{aligned} \tag{2.5}$$

If $(p_A, p_B) \in \mathscr{D}_2$, *then*

$$\begin{aligned} \mu_A(p_A, p_B) &= 1 - \frac{U_A p_A - U_B p_B}{(U_A - U_B) R_2} + \frac{R_1}{R_2}; \\ \mu_B(p_A, p_B) &= \frac{U_A p_A - U_B p_B}{(U_A - U_B) R_2} - \frac{R_1}{R_2}. \end{aligned} \tag{2.6}$$

If $(p_A, p_B) \in \mathscr{D}_3$, *then*

$$\begin{aligned} \mu_A(p_A, p_B) &= \text{Min} \left\{1, 1 - \frac{U_A p_A}{(U_A - U_D) R_2} + \frac{R_1}{R_2}\right\}; \\ \mu_B(p_A, p_B) &= 0. \end{aligned} \tag{2.7}$$

(*Proof in Appendix.*)

It is instructive to examine the graph of the demand function $\mu_B(p_A, p_B)$ for a fixed value of p_A — say $\bar{p}_A$, (see Figure 5), and the graph of the demand function $\mu_A(p_A, p_B)$ for a fixed value of p_B, — say $\bar{p}_B$ (see Figure 6).

Let us assume that $\bar{p}_A$ is such that $(\bar{p}_A, 0) \in \mathscr{D}_2$. As p_B increases from zero, then $\mu_B(p_A, p_B)$ decreases at a constant rate until $(\bar{p}_A, p_B)$ "enters" into $\mathscr{D}_1$, i.e., until $p_B = \pi_B(0)$. For values of $p_B > \pi_B(0)$, $\mu_B(\bar{p}_A, p_B)$ still decreases but at a faster rate until $(\bar{p}_A, p_B)$ enters into $\mathscr{D}_3$, where $\mu_B(\bar{p}_A, p_B)$ remains equal to zero. Consequently, the demand function of duopolist B, at fixed $\bar{p}_A$, has a "kink" at $p_B = \pi_B(0)$. This kink is easily understandable if we notice that a p_B-price cut in $\mathscr{D}_1$ not only increases duopolist's B *share*, but also increases the *size* of the market; in $\mathscr{D}_2$, however, a p_B-price cut only increases the share, without changing the size of the market, since it is already fully served.

Now we consider variations of μ_A at fixed $\bar{p}_B$, assuming that $\bar{p}_B > \pi_B(0)$. A simple examination of the analytic expression of the demand function μ_A gives the following diagram. Consequently, the demand function of duo-

[3] These three sets correspond to the three price regions identified in our heuristic introduction.

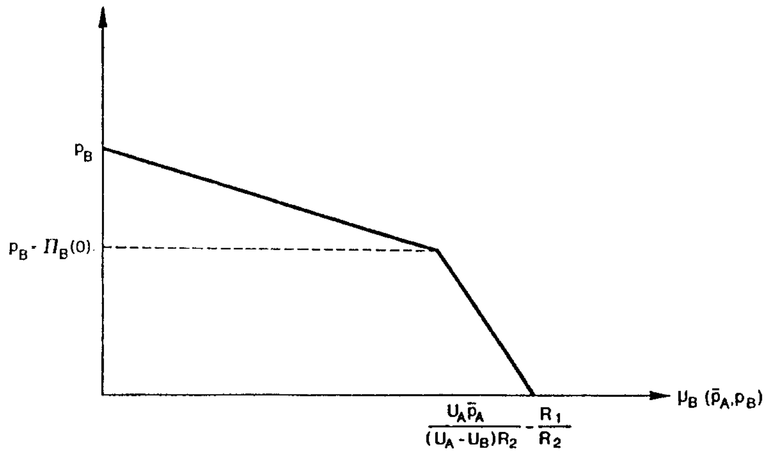

FIGURE 5

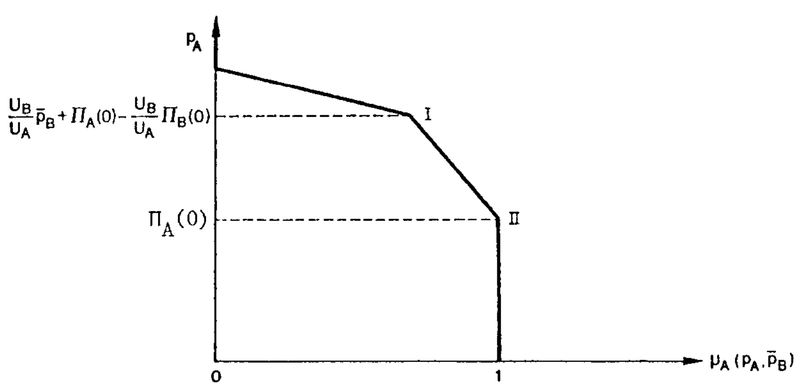

FIGURE 6

polist A at fixed $\bar{p}_B$ has two kinks: kink I corresponds to the value of p_A for which duopolist B is out of the market; kink II corresponds to the value of p_A for which duopolist A serves the whole market alone.

We close this subsection with two definitions.

The *profit function* P_A (resp. P_B) of duopolist A (resp. duopolist B) is the real-valued function defined on $S_A \times S_B$ by $P_A(p_A, p_B) = p_A \cdot \mu_A(p_A, p_B)$ (resp. $P_B(p_A, p_B) = p_B \cdot \mu_B(p_A, p_B)$).

A *Cournot equilibrium point* is a pair of price strategies (p_A^*, p_B^*) such that, for all $p_A \in S_A$, $P_A(p_A, p_B^*) \leqslant P_A(p_A^*, p_B^*)$ and, for all $p_B \in S_B$, $P_B(p_A^*, p_B) \leqslant P_B(p_A^*, p_B^*)$ and $\mu_A(p_A^*, p_B^*) + \mu_B(p_A^*, p_B^*) \leqslant 1$.

2.b *Static Theory: The Equilibrium Points of the Game* (U_A, U_B, R_1, R_2)

We are now in a position to answer the first issue stated in our introduction: in which price region $\mathscr{D}_1$ the Cournot equilibrium point will fall? Our main results in static theory consist in determining exactly the set of parametric values (U_A, U_B, R_1, R_2) which lead the Cournot equilibrium point in region $\mathscr{D}_1$, $\mathscr{D}_2$ or $\mathscr{D}_3$, respectively; furthermore the exact analytic values of equilibrium prices are derived in each case. This is formally stated in the following propositions.

PROPOSITION 1. $(p_A^*, p_B^*) \in \mathscr{D}_1 \Leftrightarrow (U_A, U_B, R_1, R_2)$ *verifies*

$$\frac{R_1}{R_2} < \frac{1}{3}\frac{U_A - U_B}{U_A - U_0}. \tag{2.8}$$

Furthermore, (p_A^*, p_B^*) *is unique and defined by*

$$p_A^* = \frac{2(U_A - U_0)(U_A - U_B)(R_1 + R_2)}{U_A(4U_A - U_B - 3U_0)},$$
$$p_B^* = \frac{(U_B - U_0)(U_A - U_B)(R_1 + R_2)}{U_B(4U_A - U_B - 3U_0)}. \tag{2.9}$$

PROPOSITION 2. $(p_A^*, p_B^*) \in \mathscr{D}_2 \Leftrightarrow (U_A, U_B, R_1, R_2)$ *verifies*

$$\frac{1}{3}\frac{U_A - U_B}{U_A - U_0} \leqslant \frac{R_1}{R_2} \leqslant 1. \tag{2.10}$$

Furthermore, (p_A^*, p_B^*) *is unique and defined by*

$$p_A^* = \frac{(U_A - U_B)\,R_2 + (U_A - U_0)\,R_1}{2U_A},$$
$$p_B^* = \frac{U_B - U_0}{U_B} \tag{2.11}$$

if

$$\frac{1}{3}\frac{U_A - U_B}{U_A - U_0} \leqslant \frac{R_1}{R_2} < \frac{U_A - U_B}{U_A + 2U_B - 3U_0}; \tag{2.12}$$

and by

$$p_A^* = \frac{(U_A - U_B)(2R_2 + R_1)}{3U_A},$$
$$p_B^* = \frac{(U_A - U_B)(R_2 - R_1)}{3U_B}. \tag{2.13}$$

if

$$\frac{U_A - U_B}{U_A + 2U_B - 3U_0} \leqslant \frac{R_1}{R_2} \leqslant 1. \tag{2.14}$$

PROPOSITION 3. $(p_A^*, p_B^*) \in \mathscr{D}_3 \Leftrightarrow (U_A, U_B, R_1, R_2)$ *verifies*

$$\frac{R_1}{R_2} > 1. \tag{2.15}$$

Furthermore, (p_A^*, p_B^*) *is unique and defined by*

$$p_A^* = \pi_A(0) - \frac{U_B}{U_A}\pi_B(0), \tag{2.16}$$

$$p_B^* = 0.$$

We do not provide here with proofs of these propositions, but only with hints which could serve as guidelines for these proofs.[4] For propositions 1 and 2, the basic principle of proof can be described as follows. Since the demand functions are piecewise linear, profit functions are piecewise quadratic. Then equilibrium prices can be obtained by solving the first order conditions for maximization of p_A and p_B. For the solutions so obtained to fall in $\mathscr{D}_1$ (resp. $\mathscr{D}_2$), it is necessary and sufficient that condition (2.8) (resp. (2.10)) be verified. We know that these solutions are mutual best replies in the restricted domain $\mathscr{D}_1$ (resp. $\mathscr{D}_2$). But this is not sufficient to guarantee that they are mutual best replies on the *whole* domain of strategies. To illustrate, suppose, for instance, that (p_A^*, p_B^*) are as defined by (2.9); then we know that p_A^* beats any strategy p_A in the projection of $\mathscr{D}_1$ against p_B^*. But it remains to show that p_A^* also beats any strategy in the projection of $\mathscr{D}_2 \cup \mathscr{D}_3$ against p_B^*. Showing that is almost only a matter of calculation; it would be tedious and casuistic to explicitate the whole argument.

As for proposition 3, the proof goes as follows. First notice that no equilibrium pair of prices (p_A^*, p_B^*) with $\mu_B(p_A^*, p_B^*) > 0$ could exist with $\mu_A + \mu_B \leqslant 1$ if condition $R_1/R_2 \leqslant 1$ is not satisfied. Accordingly, $R_1/R_2 > 1$ means that if (p_A^*, p_B^*) is an equilibrium pair of prices, then it should be in $\mathscr{D}_3$. A simple argument which is left to the reader shows that the pair of prices defined as in (2.16) is an equilibrium point.

Let us now briefly comment on these propositions. First they provide us with a full characterization of the non-cooperative market structure as a function of the parameters which specify the quality of the products and the income distribution. The condition $R_1/R_2 \gtrless 1$ is essential with this respect. If $R_1/R_2 < 1$, then both the duopolists are in the market; in the opposite

[4] The interested reader may obtain the complete proof on request to the authors.

case, only duopolist A remains in the market. The condition $R_1/R_2 > 1$ means that the utility of the income of the richest customer does not exceed twice the utility of the income of the poorest one. Even though this condition is particular to our specific example, it points out that *income differentiation plays a central role to sustain product differentiation.* Of course, this role would even be enhanced, should the products be realized under different cost conditions, which is normally the case for a high quality product and a standard one. In that case, higher costs would be coupled with the more sophisticated model, which in turn would reinforce price differences. Probably more income inequality would be needed to justify the survival of two markets: the first one where a high quality product is sold at a high price to the rich people, and the other where a standard product is sold at a lower price to the poor ones.

A difficult question arises in this context: is it better, in welfare terms, to maintain both products through a sufficient income differentiation, or to enforce the existence of only one of them through levelling of income? An answer to this question for our specific example will be given in the next subsection. Propositions 1, 2 and 3 state that the Cournot equilibrium point is unique in $\mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$. Interestingly, the strategy used by duopolist B at the equilibrium point $(\pi_A(0) - (U_B/U_A)\,\pi_B(0), 0)$ in $\mathscr{D}_3$ is the one which "hurts" duopolist A the most seriously in the whole set of strategies for B. This can be seen as follows. If $R_1/R_2 > 1$, then for any price p_B^*, duopolist A can always enforce duopolist B to be out of the market by choosing $p_A^* = (U_B/U_A)\,p_B^* + \pi_A(0) - (U_B/U_A)\,\pi_B(0)$, and then realize a profit equal to $p_A^* \cdot 1 = (U_B/U_A)\,p_B^* + \pi_A(0) - (U_B/U_A)\,\pi_B(0)$. Clearly this profit is made the smallest possible if duopolist B chooses $p_B^* = 0$. Then duopolist 2 must choose $p_L^* =^{\mathrm{Def}} \pi_A(0) - (U_B/U_A)\,\pi_B(0)$, which corresponds to the limit pricing practice, as suggested in Modigliani [6]: this price is the highest price that duopolist A can set, continuing to guarantee himself that duopolist 1 could not enter the market. *Hence, the limit price is viewed here as an equilibrium strategy which endows it with a game theoretic stability content.*

The final outcome of our propositions is to provide us with the explicit equilibrium price values, as functions of the parameters (U_A, U_B, R_1, R_2): The next subsection is devoted to the propositions of comparative statics which can be derived from parametric variations.

2.c *Comparative Statics*: *Variations of the Equilibrium Points*

As it is now clear from the preceding propositions, the equilibrium price values p_A^* and p_B^*, or the limit price p_L^* (if $R_1/R_2 > 1$), move according to the basic parameters of the game. Variations of R_1 or R_2 amount to variations in the income distribution.

Two types of variations in the income distribution can be considered, and their impact on the equilibrium points of the game analyzed. First we may examine the impact of a change in the *mean* m of the distribution while maintaining its standard deviation σ constant. Such a change leads the income of each customer to be increased by an identical amount. Alternatively, one may be interested to analyze the impact, on equilibrium prices, of a reallocation among the customers of the *same* total mass of income. If this reallocation leads to a more inegalitarian distribution, this change implies an increase in the *standard deviation* σ of the distribution, while maintaining its mean constant. A simple calculation shows that $m = R_1 + \frac{1}{2}R_2$ and $\sigma = R_2/2\sqrt{3}$, so that $R_1/R_2 = m/(2\sqrt{3}\cdot\sigma) - \frac{1}{2}$.

Substituting m by its value in all the equilibrium values for (p_A^*, p_B^*) and maintaining σ(or R_2) constant, the following property is easily derived:

(i) p_A^* and p_B^* are piecewise linear functions of m;

(ii) if $\dfrac{R_1}{R_2} \leqslant \dfrac{U_A - U_B}{U_A + 2U_B - 3U_0}$, $\left.\dfrac{dp_B^*}{dm}\right|_{R_2=c^t} > 0$ and $\left.\dfrac{dp_A^*}{dm}\right|_{R_2=c^t} > 0$;

(iii) if $\dfrac{U_A - U_B}{U_A + 2U_B - 3U_0} \leqslant \dfrac{R_1}{R_2} \leqslant 1$, $\left.\dfrac{dp_B^*}{dm}\right|_{R_2=c^t} < 0$ and $\left.\dfrac{dp_A^*}{dm}\right|_{R_2=c^t} > 0$;

(iv) if $\dfrac{R_1}{R_2} > 1$, $\dfrac{dp_L^*}{dm} > 0$.

Figure 7 depicts $(p_A^* - p_B^*)$-variations as functions of m.

Accordingly, at low average income levels, *both firms have incentives to support a policy which aims at increasing average income* since such an increase is profitable to both of them; *beyond some level, such an increase is no longer profitable to duopolist B*: there, more and more customers are rich enough to abandon the standard product for the high quality one.

Substituting σ by its value in all the equilibrium values for p_A^* and p_B^* and maintaining m constant through R_2, the following property is easily derived:

(i) p_A^* and p_B^* are piecewise linear functions of σ;

(ii) if $\dfrac{R_1}{R_2} < \dfrac{1}{3}\dfrac{U_A - U_B}{U_A - U_0}$ or if $\dfrac{U_A - U_B}{U_A + 2U_B - 3U_0} < \dfrac{R_1}{R_2} < 1$,

$$\left.\frac{dp_B^*}{d\sigma}\right|_{m=c^t} > 0 \text{ and } \left.\frac{dp_A^*}{d\sigma}\right|_{m=c^t} > 0;$$

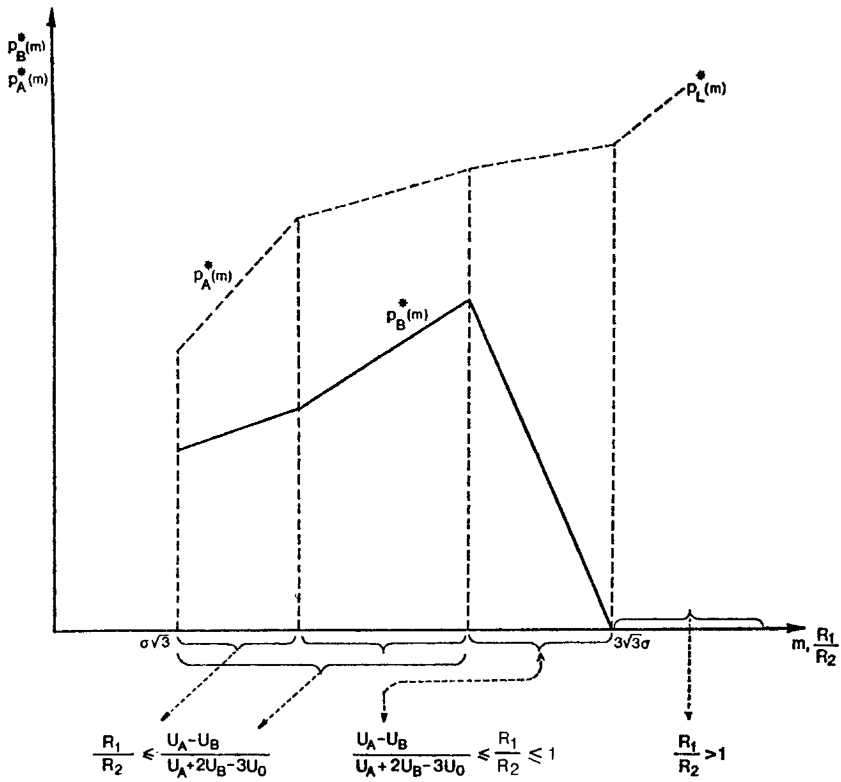

FIGURE 7

(iii) if $\frac{1}{3}\frac{U_A - U_B}{U_A - U_0} < \frac{R_1}{R_2} < \frac{U_A - U_B}{U_A + 2U_B - 3U_0}$,

$$\left.\frac{dp_B^*}{d\sigma}\right|_{m=c^t} < 0 \text{ and } \left.\frac{dp_A^*}{d\sigma}\right|_{m=c^t} \gtreqless 0 \Leftrightarrow U_A - U_B \gtreqless U_B - U_0;$$

(iv) if $\frac{R_1}{R_2} > 1$, $\frac{dp_L^*}{d\sigma} < 0$.

Figure 8 depicts (p^*, p^*)-variations as functions of σ.

First we notice again that a systematic reduction of income inequality must enforce product B to disappear from the market ($\sigma \leqslant \hat{\sigma}$). Second, the oscillating behaviour of equilibrium prices w.r.t. σ-variations is really surprising; in particular, if $U_A - U_B < U_B - U_0$, both prices have to go up in the domain defined by (2.12) with a reduction of income inequality! Finally, we may now determine, as announced in subsection 2.b, whether it is better in welfare terms to maintain, or not, both products through a sufficient income differen-

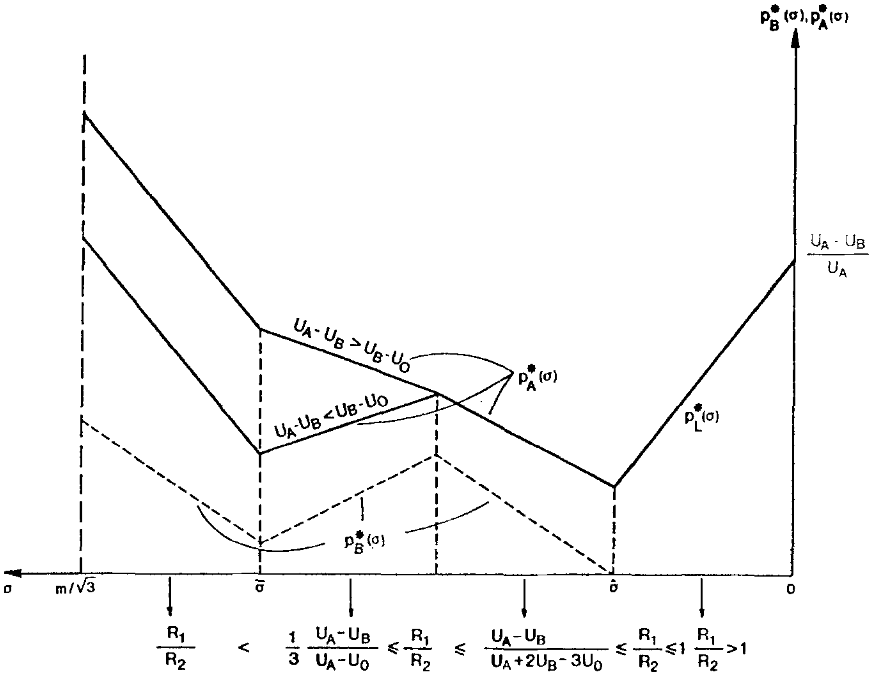

FIGURE 8

tiation. To this end, let us determine the highest total utility reached by the customers over all values of σ. Since with a constant mean m, the total utility of income $\int_0^1 U(0; R(t))\,dt$ is invariant w.r.t. variations in σ, this reduces to compare the total utility reached at $\bar{\sigma}$ with the total utility reached at $\hat{\sigma}$: all other σ-values lead to higher prices for both the products and thus should not have to be considered. Furthermore, since $p_B^*(\bar{\sigma}) > 0 = p_B^*(\hat{\sigma})$, and since at $\hat{\sigma}$, all the customers are served with the high quality product, $\hat{\sigma}$ is certainly preferred to $\bar{\sigma}$ in welfare terms if $p_A^*(\bar{\sigma}) > p_A^*(\hat{\sigma}) = p_L^*(\hat{\sigma})$. A simple calculation shows indeed that this inequality must hold. Consequently, *the optimal welfare solution requires a degree of income differentiation which is just sufficient to guarantee that the low quality product B cannot be sold on the market.* For this degree of income differentiation, the seller of product A will practice his limit price $p_L^*(\hat{\sigma})$, which is the lowest price which can be hoped to be observed on a non-cooperative market.

Now, let us examine variations in the parameters U_A and U_B.

If the duopolists are allowed to change the quality characteristics which surround their products through objective or subjective improvements, it will entail variations of the parameters U_A or U_B which are derived from the underlying preferences of the customers. These latter variations will in turn modify individual reservation prices and, accordingly, the equilibrium points of the game. First consider that duopolist A improves his product. Then we derive:

(i) if (2.8) or (2.14) holds, $\frac{dp_B^*}{dU_A} > 0$ and $\frac{dp_A^*}{dU_A} > 0$;

(ii) if (2.12) holds, $\frac{dp_B^*}{dU_A} = 0$ and $\frac{dp_A^*}{dU_A} > 0$.

If duopolist B improves his product, we obtain:

(i) if (2.8) holds, $\frac{dp_B^*}{dU_B}$? and $\frac{dp_A^*}{dU_B} < 0$;

(ii) if (2.12) holds, $\frac{dp_B^*}{dU_B} > 0$ and $\frac{dp_A^*}{dU_B} < 0$;

(iii) if (2.14) holds, $\frac{dp_B^*}{dU_B} < 0$ and $\frac{dp_A^*}{dU_B} < 0$.

Thus, an improvement of product A always leads to an increase in both prices and an improvement of product B always leads to a decrease of p_A^*. Furthermore, if (2.14) is verified, *an increase in the quality of the standard product B must lead to a decrease of its price*! What is still more surprising is the fact that, in the previous case, an increase in the quality of product B not only leads to a decrease of its price, but also to *a decrease of duopolist B's profits*. It is indeed easily calculated that, under condition (2.14), duopolist B's market share $\mu_B(p_A^*, p_B^*)$ is equal to $(R_2 - R_1)/3R_2$, which is invariant w.r.t. U_A. Finally, notice that *the oscillating behaviour of* p_B^* *w.r.t. quality thus implies that no general statement can be inferred about the dependence of prices on quality.*

Not only $\mu_B(p_A^*, p_B^*)$ is invariant w.r.t. quality parameters under condition (2.14) but so is also $\mu_A(p_A^*, p_B^*)$ which is there equal to $(2R_2 + R_1)/3R_2$. This invariance entails another important implication when combined with the corresponding price variations. It has just been shown that an increase in the quality of A *increases* both prices in the domain defined by (2.14), while an increase in the quality of B has the opposite effect. Since in this domain, market shares are invariant w.r.t. quality variations, the higher the quality of product A and the lower the quality of the standard product (above U_0), the larger will be the profits of both duopolists.

We thus have the two following conclusions. First, as far as the duopolists are considering to choose the quality of the products they sell in the domain defined by (2.14), there is an incentive for both of them to maximize product differentiation there. Second, either there are incentives for both duopolists to maintain the parameters U_A and U_B so that the game is defined by the domain

$$0 \leqslant \frac{R_1}{R_2} < \frac{U_A - U_B}{U_A + 2U_B - 3U_0},$$

which is impossible without a sufficient degree of product differentiation (if U_A is "too" close to U_B, the above domain is almost empty); or the parameters U_A and U_B are such that the game falls in the domain defined by

$$\frac{U_A - U_B}{U_A + 2U_B - 3U_0} \leqslant \frac{R_1}{R_2} \leqslant 1,$$

in which case there is an incentive for maximal product differentiation there. In both cases, *we have thus natural forces which lead the duopolists to maintain product differentiation, contrary to the usual inference from Hotelling's model in location theory.*

Finally another by-product of our parametric variations analysis is connected to the so-called "Bertrand's problem". It was argued by Bertrand [1] that if two duopolists sell an *homogeneous* product to a continuum of customers, then the competitive price will prevail, with indeterminacy on the sharing of the market. Bertrand's market situation may be viewed in our picture as the limit of a sequence of markets with two duopolists selling products A and B, the quality characteristics of which become closer and closer. Clearly, if B remains fixed and if product A "tends" to product B while remaining always preferred to B, this sequence of markets will entail a sequence $\{U_A(n)\}$ of successive parametric values for product A converging to U_B in such a way that, for all n, $U_A(n) - U_B > 0$. Assuming that $R_1/R_2 \leqslant 1$, the sequence $\{U_A(n)\}$ will "enter" from some rank N in the domain defined by (2.14), and remain there for all $n \geqslant N$. It is then easily verified that the corresponding sequences of equilibrium prices $\{p_A^*(n)\}$ and $\{p_B^*(n)\}$ both tend to zero, that is, to the competitive price (recall that the duopolists produce A and B at no cost). Furthermore, since for all $n \geqslant N$, the market shares remain invariant w.r.t. quality variations, *a unique solution to Bertrand's problem is provided by our analysis, for any market situation where a small dissimilitude exists between the products, which maintains the objective preference for product A over product B.*

3. Conclusion

In spite of its attractive features, the preceding analysis should not delude us: the above model is very specific and hardly simplifies the discussion of basic issues which are still challenged in the theory of imperfect competition.

Probably, the level of generality of our model compares with most of the existing literature on product differentiation. Thus, many of these models assume that consumers have identical income and uniform distribution of singlepeaked preferences (see, e.g., [5] or [8]). We have opted for an analogous approach but where income disparities have been priviliged against tastes differentiation. But the basic issues encountered in past attempts to generalize these contributions are not removed if, instead, our approach is considered.

A first issue is the problem of existence of a Cournot equilibrium. So far there have been few encouraging results in this direction, and a Cournot equilibrium has been shown to exist in particular cases only (our model is a new example in this class). In fact, there are two sources of difficulties when dealing with existence. First, the use of some firms strategies entails discontinuities in the demand curves of the firm: it may happen that if a competitor lowers his price slightly below a given level, he gains the entire market ("undercutting" strategies). Such discontinuities are typical in location models "à la Hotelling" and it has been proved elsewhere that, even for the simplest location models, these discontinuities may entail non-existence [3]. The model proposed here does not share this difficulty. Indeed, by contrast with Hotelling's contribution, no undercutting strategy exists in our case: When price is moved down, the customers move gradually from one competitor to the other. Of course, this "smoothness" directly follows from the fact that the income distribution itself is smooth. Should there exist "atoms" in the latter distribution, then all customers with identical income would switch at the same price from one competitor to the other, entailing again a discontinuity in the demand curve.[5] As the proof of existence of a Cournot equilibrium would normally require the continuity of the demand function, it could be interesting to study the class of income distributions which would preserve this continuity. A reasonable conjecture seems to be that if the customers distribution has no atom in the space of preferences and income distributions, then the demand function should be continuous.

Unfortunately, the second source of difficulty seems much harder to be dealt with. It arises from the fact that fixed point arguments used in the existence proofs are based on quasi-concavity of the profit functions. However there is no general reason why profit functions should be quasi-concave. Roberts and Sonnenschein [7] have exhibited counterexamples to existence in a simple general equilibrium context, based on the fact that, if income effects appear in the consumers demand functions, profit functions of duopolists need not be quasi-concave. Similarly, the authors have unpublished

[5] Yet this conclusion would only hold if all customers have also identical tastes for, otherwise, taste differentiation could also guarantee the absence of demand discontinuities. It is interesting to notice that our model could be interpreted in that way by assuming that all customers have identical income, but varying tastes.

examples in location theory where, in spite of the continuity of the demand functions, equilibrium does not exist for the same reason [3]. If the problem analyzed in the present paper would be cast in more general terms, analogous difficulties will probably reappear. One possible way to escape them would require the use of mixed strategies [2]. But, beyond the difficulty of calculating equilibrium mixed strategies over a set of continuous pure strategies, their use seems to involve difficult problems of interpretation. In the view of the authors, it is still useful to develop further particular models in the field, and study their analogies and differences. Hopefully "some" theory of imperfect competition could emerge from these comparisons, even if no general theory becomes available in the short run.

As long as the problem of existence remains unsolved, it seems of little interest to discuss the issues of uniqueness and stability.

More interesting would be to consider a natural extension of our analysis to more than two oligopolists. It is also a basic issue in the theory of imperfect competition as to whether the perfectly competitive paradigm is the "limit" of imperfect competition. This question has almost always been analyzed for markets where an homogeneous product is sold by an increasing number of competitors.[6] A reasonable conjecture is that with an increasing number of *substitutes* in a given industry, more competition will occur, and equilibrium prices will have to go down, as it is the case when the number of oligopolists increases on a market for an homogeneous product. Here the analysis looks more encouraging: the authors have obtained that this conjecture indeed verified in an extended version of the present model.

Other basic issues have not been dealt in the present paper, like the role of cost structures on equilibrium, or comparison with alternative market structures, like collusion or monopoly. Clearly we only did a little walk in a fascinating country, the exploration of which will hopefully be pursued by the authors.

APPENDIX: *Proof of Lemma* 2

Let

$$t_A(p_A) = \frac{U_A p_A}{(U_A - U_0) R_2} - \frac{R_1}{R_2}, \quad t_B(p_B) = \frac{U_B p_B}{(U_B - U_0) R_2} - \frac{R_1}{R_2}$$

and

$$\tilde{t}(p_A, p_B) = \frac{U_A p_A - U_B p_B}{(U_A - U_B) R_2} - \frac{R_1}{R_2}.$$

[6] A notable exception in a general equilibrium context is provided by the work of O. Hart [4].

We first consider (2.5) and (2.6). For the proof we have to show:

$$\mu_A(p_A, p_B) = 1 - \bar{t}(p_A, p_B) \text{ and } \mu_B(p_A, p_B)$$
$$= \begin{cases} \bar{t}(p_A, p_B) - t_B(p_B) & \text{for } (p_A, p_B) \in \mathscr{D}_1 \\ \bar{t}(p_A, p_B) & \text{for } (p_A, p_B) \in \mathscr{D}_2 . \end{cases}$$

The steps of the proof are as follows. First, t_A, t_B and $\bar{t}$ are all in [0, 1[, for otherwise μ_A or μ_B would be equal to zero, a contradiction to the fact that $(p_A, p_B) \in \mathscr{D}_1 \cup \mathscr{D}_2$. Second, $t_B < \bar{t}$ without which μ_B would be equal to zero, which also contradicts $(p_A, p_B) \in \mathscr{D}_1 \cup \mathscr{D}_2$. Third, $t_A < \bar{t}$. Suppose on the contrary $\bar{t} \leqslant t_A$. Then we have

$$\frac{U_A p_A - U_B p_B}{U_A - U_B} \leqslant \frac{U_A p_A}{U_A - U_0}.$$

However $\bar{t} > t_B$ amounts to

$$\frac{U_A p_A - U_B p_B}{U_A - U_B} > \frac{U_A p_A}{U_A - U_0}.$$

Given that $\bar{t} > t_B$ we thus arrive at a contradiction. Fourth, $t_B > 0$ if $(p_A, p_B) \in \mathscr{D}_1$. Indeed, assuming $t_B \leqslant 0$ would imply $\mu_B = \hat{t}$ and, consequently, $\mu_A + \mu_B = 1$ which contradicts $(p_A, p_B) \in \mathscr{D}_1$. Fifth, and last, $t_B \leqslant 0$ when $(p_A, p_B) \in \mathscr{D}_2$. Indeed, with $t_B > 0$ we would have $\mu_A + \mu_B = 1 - t_B < 1$, a contradiction. Given the above inequalities, it is then easy to obtain μ_A and μ_B.

Let us now prove (2.7). As $\mu_B = 0$, we have $t_A \subsetneq t_B$ so that $\mu_A = 1 - t_A$ and (2.7) follows.

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