A Non-cooperative Equilibrium for Supergames
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A Non-cooperative Equilibrium for Supergames [1, 2]

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I. INTRODUCTION

John Nash has contributed to game theory and economics two solution concepts for nonconstant sum games. One, the non-cooperative solution [9] is a generalization of the minimax theorem for two person zero sum games and of the Cournot solution; and the other, the cooperative solution [10], is completely new. It is the purpose of this paper to present a non-cooperative equilibrium concept, applicable to supergames, which fits the Nash (non-cooperative) definition and also has some features resembling the Nash cooperative solution. " Supergame " describes the playing of an infinite sequence of "ordinary games" over time.[3] Oligopoly may profitably be viewed as a supergame. In each time period the players are in a game, and they know they will be in similar games with the same other players in future periods.

The most novel element of the present paper is in the introduction of a completely new concept of solution for non-cooperative supergames. In addition to proposing this solution, a proof of its existence is given. It is also argued that the usual notions of " threat " which are found in the literature of game theory make no sense in non-cooperative supergames. There is something analogous to threat, called " temptation ", which does have an intuitive appeal and is related to the solution which is proposed.

In section II the ordinary game will be described, the non-cooperative equilibrium defined and its existence established. Section III contains a description of supergames and supergame strategies. In section IV a definition and discussion of a non-cooperative equilibrium for supergames is given. This equilibrium shares some of the attributes of the Nash-Harsanyi [10, 6] cooperative solution, and is very much in the spirit of the solution proposed several years ago by Professor Robert L. Bishop in the *American Economic Review* [2]. In section V existence will be proved, in section VI some assumptions will be relaxed, and in section VII economic applications will be discussed.

II. THE GAME AND THE NASH NON-COOPERATIVE EQUILIBRIUM

An " ordinary game " is a game in which each player has a set of strategies which is a compact, convex subset of a Euclidean space of finite dimension, there are a finite number of players and the payoff to each player is a function of the chosen strategies of all players. In this section, the ordinary game will be described in detail. A proposition, due originally to Nash [9], will be proved. It establishes the existence of a non-cooperative equilibrium for the ordinary game. Although this result was previously known, it is included for completeness. A game is said to be " non-cooperative " if it is not possible for the players to form coalitions or make agreements.

1 *First version received, May* 1969; *final version received March* 1970 (*Eds*).
2 The author gratefully acknowledges the support of the National Science Foundation in the research reported here.
3 Some discussion of early work on supergames may be found in Luce and Raiffa [8], and some interesting developments in cooperative supergames, by Aumann, is begun in [1].

Denote the strategy of the ith player by s_i, a vector in r_i-dimensional Euclidean space, R^{r_i}. The strategy set of the ith player is taken to be a compact, convex subset of R^{r_i}, denoted S_i. There is a fixed, finite number of players, n, and the strategy set of the game S, is $S_1 \times \ldots \times S_n$, the Cartesian product of the individual strategy sets. A vector of strategies, one for each player, is denoted $s = (s_1, \ldots, s_n)$ and $\bar{s}_i$ denotes the strategy vector $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Thus $\bar{s}_i$ consists of the strategy choices of all players except the ith, and $s = (\bar{s}_i, s_i)$.

Payoff to the ith player is a real valued function of strategy, s, and is denoted $\pi_i(s)$. A vector of payoffs, associated with a given vector of strategies, may be denoted

$$\pi(s) = (\pi_1(s), \ldots, \pi_n(s)) \in R^n.$$

Assumptions made on the strategy space S and the payoff functions are:

A1 S_i is compact and convex $(i = 1, \ldots, n)$;

A2 The payoff functions, $\pi_i(s)$, are continuous and bounded on S, for all i;

A3 The payoff functions $\pi_i(s) = \pi_i(\bar{s}_i, s_i)$ are quasi-concave functions of s_i, for all i.

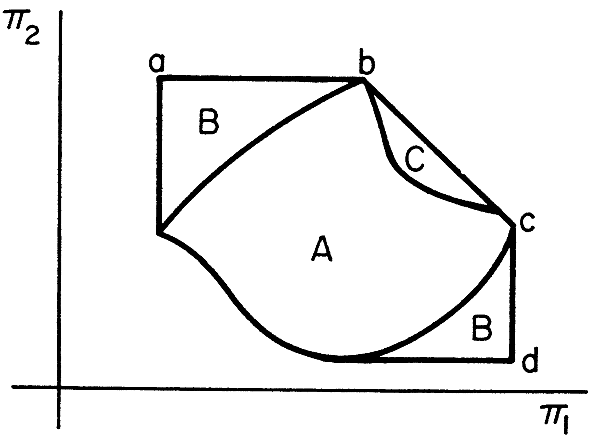

FIGURE 1

A point in the payoff space $(\pi_1(s^*), \ldots, \pi_n(s^*)) = \pi(s^*)$ is said to be "Pareto optimal" if

(i) $s^* \in S$ and
(ii) there is no $s \in S$ for which $\pi_i(s) > \pi_i(s^*)$ $(i = 1, \ldots, n)$.

Denote by H, the set of attainable payoffs: $H = \{\pi(s) | s \in S\}$. Denote by $H^* \subset H$, the set of Pareto optimal payoffs.

A4 If $\pi' \leqq \pi''$ (i.e., $\pi'_i \leqq \pi''_i$, $i = 1, \ldots, n$) and $\pi', \pi'' \in H$, then $\pi \in H$ where $\pi' \leqq \pi \leqq \pi''$;

A5 H^* is concave.

Most of the assumptions above are both reasonable and clear. The least so is A4, which will be discussed in section VI. Fig. 1 illustrates the meaning of certain of the assumptions. Region "A" is an arbitrary compact set. Compactness is required by A1 and A2. By A4, the regions denoted "B" are added, and by A5, region "C" is added. H^* is the heavy outer boundary *abcd*. Assumptions A4 and A5 mean that sets $H(\pi') = \{\pi \mid \pi > \pi', \pi \in H\}$ are convex, and for any $\pi \in H$ any non-negatively sloped ray through π will intersect H^* at exactly one point. This property will prove convenient in sections IV and V.

It remains in this section to define " non-cooperative equilibrium ", and prove its existence for ordinary games of the sort under study in this section. s^* is a non-cooperative equilibrium strategy vector if $s^* \in S$ and

$$\pi_i(s^*) = \max_{s_i \in S_i} \pi_i(\bar{s}_i^*, s_i), \quad i = 1, \ldots, n.$$

Proposition 1. *Any game satisfying A1, A2 and A3 has a non-cooperative equilibrium.*

Proof.[1] Define $\mu(s) = \mu_1(\bar{s}_1) \times \mu_2(\bar{s}_2) \times \ldots \times \mu_n(\bar{s}_n)$ as follows:

$$\mu_i(\bar{s}_i) = \{t_i \mid t_i \in S_i, \pi_i(\bar{s}_i, t_i) = \max_{s_i \in S_i} \pi_i(\bar{s}_i, s_i)\}, \quad (i = 1, \ldots, n), \quad s \in S;$$

$\mu_i(\bar{s}_i)$ is clearly compact and convex. As $\mu_i(\bar{s}_i) \subset S_i$, it is bounded. That $\mu_i(\bar{s}_i)$ is closed follows from the continuity of π_i, and convexity follows from the quasi-concavity of π_i in s_i. As this holds for all i, the sets $\mu(s) = \mu_i(\bar{s}_i) \times \mu_2(\bar{s}_2) \times \ldots \times \mu_n(\bar{s}_n)$ are compact, convex and subsets of S.

If it can be shown that the correspondence μ: $S \rightarrow S$ is upper semi-continuous, the Kakutani [7] fixed point theorem may be applied. A fixed point of μ is a non-cooperative equilibrium. Let $\bar{s}_i^l \in \bar{S}_i$, $l = 1, 2, \ldots$, be a sequence of strategies converging to $\bar{s}_i^0$. $\mu_i(\bar{s}_i)$ is upper semi-continuous if, when

(*a*) $s_i^l \in \mu_i(\bar{s}_i^l)$, $\quad l = 1, 2, \ldots,$ and

(*b*) $\lim_{l \rightarrow \infty} \bar{s}_i^l = \bar{s}_i^0$ and $\lim_{l \rightarrow \infty} s_i^l = s_i^0$ then

(*c*) $s_1^0 \in \mu_i(\bar{s}_i^0)$.

Assume a sequence as described in (*a*) and (*b*), but assume (*c*) is false (i.e., $s_i^0 \notin \mu_i(\bar{s}_i^0)$). If $s_i^0 \notin \mu_i(\bar{s}_i^0)$, then $\pi_i(\bar{s}_i^0, s_i^0) < \pi_i(\bar{s}_i^0, s_i')$ for $s_i' \in \mu_i(\bar{s}_l^0)$. Say $\pi_i(\bar{s}_i^0, s_i') - \pi_i(\bar{s}_i^0, s_i^0) = \varepsilon > 0$. Now consider $\pi_i(\bar{s}_i^l, s_i^l)$. By the continuity of π_i, it is possible to choose an arbitrary $\delta > 0$ such that for $l \geqq \mathscr{L}(\delta)$ ($\mathscr{L}(\delta)$, finite),

$$\pi_i(\bar{s}_i^0, s_i') - \delta < \pi_i(\bar{s}_i^l, s_i^l) < \pi_i(\bar{s}_i^0, s_i') + \delta.$$

Choosing $\delta < \varepsilon$ leads to a contradiction; hence, $s_i^0 \in \mu_i(\bar{s}_i^0)$ and μ_i is upper semi-continuous. As this holds for all i, μ is upper semi-continuous and has a fixed point. Let such a fixed point be $s^* \in \mu(s^*)$. By the definition of the μ_i,

$$\max_{s_i \in S_i} \pi_i(\bar{s}_i^*, s_i) = \pi_i(s^*), \quad i = 1, \ldots, n,$$

therefore s^* is a non-cooperative equilibrium strategy vector and $\pi(s^*)$ a non-cooperative equilibrium payoff vector.

It may be noted in passing that when the payoff functions are profit functions, the players are firms and the strategies are prices or quantities, the game is a (single period) oligopoly. The non-cooperative equilibrium becomes the same as the " Cournot solution " [3]. In this instance S_i is merely the interval of prices (or quantities) among which the firm is allowed to choose. Frank and Quandt [5] proved the existence of the Cournot equilibrium for a quantity model. Their result is, of course, a special case of Proposition 1, above, and, *a fortiori*, a special case of the theorem of Debreu [4].

Before proceeding to the next section, a final characteristic of the payoff space will be noted: let $\rho = (\rho_1, \ldots, \rho_n)$ be a vector such that $\rho_i \geqq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \rho_i = 1$.

[1] This proposition is an easy generalization of the Nash [9] theorem, which deals with S_i which are finite sets of pure strategies, together with all mixed strategies attainable from them. The proposition is, on the other hand, a special case of a theorem of Debreu [4], which, so far as the author is aware, is the most general statement of existence of non-cooperative equilibria in finite strategy spaces.

If k is a scalar, then the points $\pi(s)+k\rho$ $(-\infty<k<\infty)$ form a ray through $\pi(s)$ having non-negative slope. By A4, for any ρ there is a unique $k(\rho) \geqq 0$ such that

$$\pi(s)+k(\rho)\cdot\rho \in H^*$$

for any $s \in S$. In particular, this property holds when s is a non-cooperative equilibrium.

III. SUPERGAMES AND SUPERGAME STRATEGIES

The games of the preceding section have been dealt with in the " normal " form—the form in which there is a payoff function for each player giving his payoff as a function of a strategy vector, $\pi_i(s)$. It is convenient now to define " supergame " in extensive form; i.e., in the form in which each " move " is described. An " ordinary " game may be termed a "finite " game because the strategy sets of the players are compact and reside in a finite space.

Now consider a sequence of ordinary games with strategy sets $S_{i1}, S_{i2}, \ldots, S_{it}, \ldots$ and payoff functions $\pi_{it}(s_t)(s_t \in S_t)$; $t = 1, \ldots$; $i = 1, \ldots, n$. The tth game has $S_{1t} \times \ldots \times S_{nt} = S_{nt} = S_t$ as its strategy set and $\pi_{it}(s_t)$ $(i = 1, \ldots, n)$ as its payoff functions. A " supergame " is a game in which the tth move $(t = 1, \ldots)$ is the playing of the tth ordinary game in the sequence. At each move a payoff is received and, if the strategy sequence $s_1, \ldots, s_t, \ldots$ is played, the payoff to the ith player in the supergame is

$$\sum_{t=1}^{\infty} \alpha_{it}\pi_{it}(s_t),$$

where α_{it} is the discount parameter of the ith player in the tth time period. It is obvious that a " supergame " in which the number of moves is finite is merely a finite game; hence, attention will be restricted to supergames as defined above, which have a countably infinite number of moves.

The general definition of a supergame strategy for the ith player is as follows:

$$\begin{aligned} s_{it} &= f_{it}(s_1, \ldots, s_{t-1}), \quad t = 2, 3, \ldots, \\ &= s_{i1}, \qquad\qquad\qquad t = 1. \end{aligned}$$

$f_{it}(t = 2, 3, \ldots)$ is a sequence of functions which map all preceding ordinary game strategies of all players into the present (tth) ordinary game strategy of the ith player. As there is no past information available in the initial period, there must be a particular initial move. Then $(s_{i1}, f_{i2}, f_{i3}, \ldots)$ is a supergame strategy for the ith player. Existence of non-cooperative equilibria in the supergame is no problem. Indeed the problem is the reverse; it is easy to show existence of a large number. The principle task of this paper is to choose among these in a particular way and single out certain equilibria as being of special interest.

IV. EQUILIBRIUM STRATEGIES IN THE SUPERGAME

This section is devoted to describing a very large class of supergame strategies, to showing when members of this class are non-cooperative equilibria and to introducing a new solution concept.

The exposition will be simplified by using four additional assumptions: (A6) all constituent games of the supergame are identical, (A7) the discount parameters are the same in all periods, (A8) the ordinary game has only one non-cooperative equilibrium, and (A9) the non-cooperative equilibrium is not Pareto optimal. All of these assumptions may be removed with only minor effect on the results. This will be done in section VI.

Denote by σ_i a supergame strategy for the ith player, and denote the non-cooperative equilibrium of the ordinary game by s^c. The " Cournot strategy " is denoted σ^c and is defined by $\sigma_i^c = (s_i^c, s_i^c, \ldots)$, $(i = 1, \ldots, n)$. The Cournot strategy is the repeated choice of the non-cooperative equilibrium of the ordinary game. It is immediate that

$$\sigma^c = (\sigma_1^c, \ldots, \sigma_n^c)$$

is a non-cooperative equilibrium in the supergame. Should any single player in any periods choose moves other than s_i^c he will (by definition of s^c) reduce his payoff in those periods and leave unaffected his payoff in the periods when he still chooses s_i^c.

Now a new class of non-cooperative equilibrium supergame strategies will be specified and discussed.

Let
$$B = \{s \mid s \in S,\ \pi_i(s) > \pi_i(s^c),\quad i = 1, \ldots, n\}.$$

B consists of all ordinary game strategies which dominate the ordinary game non-cooperative equilibrium. Let $s' \in B$. Now define a strategy for the ith player, σ_i', as follows:

$$s_{i1} = s_i',$$
$$s_{it} = s_i' \text{ if } s_{j\tau} = s_j' \quad j \neq i,\ \tau = 1, \ldots, t-1,\ t = 2, 3, \ldots,$$
$$s_{it} = s_i^c \text{ otherwise.}$$

Thus, the ith player chooses s_i' in period 1 and will continue to choose s_i' indefinitely, unless someone else chooses something other than $s_j'(j \neq i)$. If any player in any period chooses $s_j \neq s_j'(j \neq i)$, then in each succeeding period the ith player chooses s_i^c. The supergame strategy vector $\sigma' = (\sigma_1', \ldots, \sigma_n')$ is a non-cooperative equilibrium if:

$$\sum_{\tau=0}^{\infty} \alpha_i^\tau \pi_i(s') > \pi_i(\bar{s}_i', t_i) + \sum_{\tau=1}^{\infty} \alpha_i^\tau \pi_i(s^c), \quad i = 1, \ldots, n,$$

or

$$\frac{\alpha_i}{1-\alpha_i}[\pi_i(s') - \pi_i(s^c)] > \pi_i(\bar{s}_i', t_i) - \pi_i(s'), \quad i = 1, \ldots, n,$$

where $t_i \in S_i$ and $\pi_i(\bar{s}_i', t_i) = \max_{s_i \in S_i} \pi_i(\bar{s}_i', s_i)$.

To see whether σ_i' is the best strategy for the ith player, given $\bar{\sigma}_i'$, consider his alternatives. One is to choose σ_i', which results in using s_i' in every period, while all other players will choose $s_j'(j \neq i)$ and the discounted payoff stream will be

$$\sum_{\tau=0}^{\infty} \alpha_i^\tau \pi_i(s') = \frac{\pi_i(s')}{1-\alpha_i}.$$

Another is to choose $s_{i1} = t_i$, and $s_{it} = s_i^c(t > 1)$. t_i will yield the maximum possible payoff in period 1 (given the other players will choose $s_j'(j \neq i)$). After period 1 all other players will revert to the Cournot strategy, so the payoff maximizing choice after period 1 is $s_{it} = s_i^c$. Any other strategy is weakly dominated by one of the two just described, when the other players are using $\sigma_j'(j \neq i)$.

Which strategy to adopt simply depends upon which discounted profit stream is the larger. I.e., if the gain in the first period of maximizing against $\bar{s}_j'$ $[\pi_i(\bar{s}_i', t_i) - \pi_i(s')]$ is less than the discounted loss from being at the Cournot point in all succeeding periods

$$\left(\frac{\alpha_i}{1-\alpha_i}[\pi_i(s') - \pi_i(s^c)]\right),$$

then σ_i' is the strategy which maximizes discounted payoff for the ith player, given that the strategy choices of the other players are $(\sigma_j', j \neq i)$.

As the discount parameter approaches one from below (discount rate falls to zero), the discounted loss from being at the Cournot point goes to infinity, while the single period gain from choosing t_i is finite and unchanging. So for any $s' \in B$ there is a lower bound for α_i, $\alpha_i(s')$, (such that $\alpha_i(s')<1$) and if $s' \in B$ and $\alpha_i>\alpha_i(s')$, then σ'_i is optimal against $\bar{\sigma}'_i$. If these conditions hold for $i = 1, \ldots, n$, then $(\sigma'_1, \ldots, \sigma'_n)$ is a non-cooperative equilibrium.

Certain of the $s \in B$ are of special interest. There is a subset $B^* \subset B$ of move vectors which give rise to Pareto optimal payoff vectors:

$s^* \in B^*$ if

(a) $s^* \in B$ and

(b) $\pi(s^*) \in H^*$.

In considering a move vector (i.e. an ordinary game strategy vector), $s^* \in B^*$, why might a player cease choosing s_i^* if he has reason to believe the others will continue choosing $s_j^*(j \neq i)$? Clearly, he may feel a temptation to choose t_i (which maximizes the single period payoff against $\bar{s}_i^*$) because of the extra payoff which may be gained in the short run $[\pi_i(\bar{s}_i^*, t_i)-\pi_i(s^*)]$. Because the players should never, in the long run, receive less than $\pi(s^c)$ per period and because they may follow strategies which send them to π^c under some circumstances, it is intuitively appealing to measure the temptation associated with s^* in relation to $\pi_i(s^*)-\pi_i(s^c)$. Associated with the equilibrium proposed in this paper is the equilibrium move vector, s^*, which satisfies:

$$s^* \in B^*, \qquad \ldots(1)$$

$$\frac{\pi_i(\bar{s}_i^*, t_i)-\pi_i(s^c)}{\pi_i(s^*)-\pi_i(s^c)} = \frac{\pi_j(\bar{s}_j^*, t_j)-\pi_j(s^c)}{\pi_j(s^*)-\pi_j(s^c)}, \quad i, j = 1, \ldots, n. \qquad \ldots(2)$$

This point is Pareto optimal and leaves each player equally tempted (in the sense of the preceding paragraph) to maximize against $\bar{s}_i^*$. An alternative way of expressing (2) is

$$\frac{\pi_i(\bar{s}_i^*, t_i)-\pi_i(s^*)}{\pi_i(s^*)-\pi_i(s^c)} = \frac{\pi_j(\bar{s}_j^*, t_j)-\pi_j(s^*)}{\pi_j(s^*)-\pi_j(s^c)}, \quad i, j = 1, \ldots, n. \qquad \ldots(2')$$

Thus, if $\frac{\alpha_i}{1-\alpha_i} > \frac{\pi_i(\bar{s}^*, t_i)-\pi_i(s^*)}{\pi_i(s^*)-\pi_i(s^c)}$, $(i = 1, \ldots, n)$, then the strategies $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ form a non-cooperative equilibrium, where σ_i^* is defined by

$$s_{i1} = s_i^*,$$

$$s_{it} = s_i^* \text{ if } s_{j\tau} = s_j^*(j \neq i),\ \tau = 1, \ldots, t-1,\ t>1,$$

$$s_{it} = s_i^c \text{ otherwise.}$$

It should be emphasized that σ^*, in addition to being a non-cooperative equilibrium, is Pareto optimal. s^c, the non-cooperative equilibrium of the basic game need not be Pareto optimal, and, as students of oligopoly theory are aware, its oligopoly counterpart, the Cournot solution, is generally *not*. It remains to show that ordinary game strategies satisfying (1) and (2) above do, in fact, exist. Before turning to that task, some comments will be made concerning properties of this concept of solution.

It is natural to ask if the proposed solution possesses any appealing properties and and also whether one might expect a cooperative solution to emerge (such as the Nash-Harsanyi [10, 6] even though the game is non-cooperative. While the Nash-Harsanyi solution applies to ordinary games, one could propose the sequence of Nash-Harsanyi solutions of the ordinary games as the solution of the supergame. The main reason for rejecting this, and other, cooperative solutions is that they rely on features of games which are peculiar to cooperative games and absent in non-cooperative. These revolve about the notion of " threat ".

It is often part of a cooperative game that the players name threat strategies and then, if they fail to come to agreement, they are forced to carry out these threat strategies. If he were not forced, a player would do better in the absence of agreement to maximize against the strategies he expects the others to use. Applying this reasoning to all players, one would expect them to choose the non-cooperative equilibrium—if they were not forced to carry out threats. This undermines the credibility of the threats.

Now consider the cooperative game from another vantage point. When a single player (or a subset forming a coalition) calculates the best payoff he can get by himself, he does so on the assumption that all other players will band together and adopt a strategy aimed at minimizing his payoff. Even in a cooperative game, this may appear an unduly costly way for the others to act; however, as a threat to coerce the player into an agreement with all other players, it has some appeal. By contrast, in the non-cooperative game coalitions are ruled out, players cannot talk and bargain with one another; hence, it is foolish to think other players wish to minimize one's own payoff. Each will want to maximize his own payoff and will not really care about payoffs to others. In other words, threats are out of place in non-cooperative games because they cannot be clearly and effectively voiced, and because they are not credible. They need not be carried out and there is no incentive to do so.

The notion of " temptation " in the supergame is slightly analogous to threat. If a player can increase his single period profit for a period or so, he may be tempted to do so, but the other players are, in response, likely to revert to a " safe " position. This is a position in which no one has any temptation to move for the sake of short term gain.

There are certain properties which one might like an equilibrium to possess:

$\alpha 1$, The solution should be unique, and always exist;
$\alpha 2$, The solution should be independent of irrelevant alternatives;
$\alpha 3$, The solution should be Pareto optimal;
$\alpha 4$, The solution should be symmetric;
$\alpha 5$, The solution should be invariant to a positive linear transformation of a payoff function;
$\alpha 6$, The solution should be a non-cooperative equilibrium.

The Nash cooperative solution satisfies $\alpha 2$-$\alpha 5$. The solution proposed here satisfies $\alpha 3$-$\alpha 6$. Properties $\alpha 3$ and $\alpha 6$ are obviously fulfilled, as is $\alpha 5$ (note that equation (2) is free of origin and scale). The meaning of $\alpha 4$ is that the solution should not depend on who is called player 1, who player 2, etc. That $\alpha 1$ is not met is obvious already, as existence depends on the discount parameter not being too small. It will be seen that if the α_i are sufficiently near one, an equilibrium must exist. Neither the present equlibrium nor the Nash-Harsanyi need be unique (except for the $N-H$ when $n = 2$).

The irrelevant alternatives assumption, $\alpha 2$, deserves special mention. Its meaning is that if you enlarge the set of available strategies, S, to a set $A \supset S$, then one of two conditions will hold: (i) the solution to the enlarged game will be the same as in the smaller game, or (ii) the solution will be a point, $y \in A$, which was not previously available ($y \notin S$). In other words the addition of new strategies cannot affect the solution unless one of the new strategies is the new solution. Thus the solution depends only on local properties of the payoff surface in the neighbourhood of the solution. This is very restrictive.

With the solution concept presented here one can well imagine $\alpha 2$ being violated. For example, enlarge the move space from S to $A = A_1 \times \ldots \times A_n$. Conceivably one or more players find that, while the old solution $s^* \in S$, is still Pareto optimal (and s^c is still the only single period non-cooperative equilibrium), the t_i do not satisfy

$$\max_{s_i \in A_i} \pi_i(\bar{s}_i^*, s_i) = \pi_i(\bar{s}_i^*, t_i).$$

Should this happen, the point, y^*, which is the new equilibrium, might be in S, although the associated t_i will not all be in S.[1] It is good that the solution offered in this paper is not restricted by $\alpha 2$.

V. EXISTENCE OF EQUILIBRIUM

The existence proof is based upon a fixed point argument which, while it guarantees existence, does not guarantee uniqueness. The fixed point argument will be used to show that points s^* exist such that

$$\frac{\pi_i(\bar{s}_i^*, t_i)-\pi_i(s^*)}{\pi_i(s^*)-\pi_i(s^c)} = \frac{\pi_j(\bar{s}_j^*, t_j)-\pi_j(s^*)}{\pi_j(s^*)-\pi_j(s^c)}, \quad (i, j = 1, ..., n).$$

A point s^* has n points $(\bar{s}_i^*, t_i)$ associated with it. The t_i are determined by

$$\pi_i(\bar{s}_i^*, t_i) = \max_{s_i \in S_i} \pi_i(\bar{s}_i^*, s_i).$$

In fact $\bar{s}_i^* \in \bar{S}_i$ is mapped into π_i. Denote this mapping ϕ_i. A preliminary result will now be proved.

Proposition 2. *The mappings ϕ_i are continuous, for all i.*

Without loss of generality, the proposition may be proved with specific reference to ϕ_1. Let $\bar{s}_1^0$ be any point in $\bar{S}_1$ and let $s_1^0 \in S_1$ be chosen so that $\pi_1(\bar{s}_1^0, s_1^0) = \phi_1(\bar{s}_1^0)$. Let $\bar{s}_1^l (l = 1, 2, ...)$ be a sequence of points in $\bar{S}_1$ such that $\bar{s}_1^l \to \bar{s}_1^0$ as $l \to \infty$. By definition of ϕ_1, there is a s_1^l associated with $\bar{s}^l$ such that $\pi_1(\bar{s}_1^l, s_1^l) = \phi_1(\bar{s}_1^l)$, $(l = 1, 2, 3, ...)$. It must now be shown that

$$\lim_{l\to\infty} \phi_1(\bar{s}_1^l) = \phi_1(\bar{s}_1^0).$$

Clearly $\lim_{l\to\infty} \phi_1(\bar{s}_1^l) \geqq \lim_{l\to\infty} \pi_1(\bar{s}_1^l, s_1^0)$. But $\lim_{l\to\infty} \pi_1(\bar{s}_1^l, s_1^0) = \pi_1(\bar{s}_1^0, s_1^0) = \phi_1(\bar{s}_1^0)$ by continuity of π_1. But if $\lim_{l\to\infty} \phi_1(\bar{s}_1^l) > \phi_1(\bar{s}_1^0)$, there would be a value of $\tilde{s}_1 = \lim_{l\to\infty} s_1^l$ such that $\pi_1(\bar{s}_1^0, \tilde{s}_1) > \pi_1(\bar{s}_1^0, s_1^0)$, due to continuity of π_1. This, of course, contradicts the definition of ϕ_1; hence the function of ϕ_1 is continuous. The same argument may be repeated for the remaining ϕ_i.

With the continuity of the ϕ_i established, it is now possible to prove the existence of a Pareto optimal move s^*, satisfying the condition

$$\frac{\pi_i(\bar{s}_i^*, t_i)-\pi_i(s^c)}{\pi_i(s^*)-\pi_i(s^c)} = \frac{\pi_j(\bar{s}_j^*, t_j)-\pi_j(s^c)}{\pi_j(s^*)-\pi_j(s^c)},$$

$\pi_i(\bar{s}_i^*, t_i) = \phi_i(\bar{s}_i^*)$.

Proposition 3. *There exists a move $s^* \in S$ such that $\pi(s^*)$ is Pareto optimal and*

$$\frac{\phi_i(\bar{s}_i^*)-\pi_i(s^c)}{\pi_i(s^*)-\pi_i(s^c)} = \frac{\phi_j(\bar{s}_j^*)-\pi_j(s^c)}{\pi_j(s^*)-\pi_j(s^c)}, \quad (i, j = 1, ..., n).$$

For any $\rho = (\rho_1, ..., \rho_n)$, $(\rho_i \geqq 0, \Sigma\rho_i = 1)$ there is one Pareto optimal point $\pi(s_\rho)$ such that

$$\frac{\pi_i(s_\rho)-\pi_i(s^c)}{\sum_{j=1}^{n} [\pi_j(s_\rho)-\pi_j(s^c)]} = \rho_i, \quad i = 1, ..., n.$$

[1] Strictly speaking, this is necessarily true if the original equilibrium s^* is unique. If s^* and y^* are both equilibria of the smaller game, it is possible that enlarging the move space eliminates s^*, leaves y^* unaffected and creates no new equilibrium points. This still violates $\alpha 2$.

The condition of Pareto optimality ensures that this mapping from points on the unit simplex, ρ, to certain Pareto optimal profit vectors (i.e. from the unit simplex to points in the closure of B^*) is one-one and onto. Now define a mapping Ω as follows:

Ω maps a point ρ on the unit simplex into δ where:

$$\delta_i = \frac{\phi_i(\bar{s}_{i\rho}) - \pi_i(s^c)}{\sum_{j=1}^{n} [\phi_j(\bar{s}_{j\rho}) - \pi_j(s^c)]} = \Omega_i(\rho), \quad i = 1, \ldots, n.$$

Clearly δ is a point on the n-dimensional unit simplex, for $\delta_i \geqq 0$ because

$$\phi_i(\bar{s}_{i\rho}) \geqq \pi_i(s_\rho) \geqq \pi_i(s^c).$$

Continuity of the ϕ_i implies continuity of Ω; therefore the Brouwer fixed point theorem may be applied. Any point, $s^* = s_\rho$, such that $\rho = \Omega(\rho)$, satisfies the conditions of the proposition.

While existence is assured, uniqueness is not. Furthermore, existence of a point s^* does not, by itself, assure existence of an equilibrium strategy vector $(\sigma_i^*, \ldots, \sigma_n^*)$, satisfying (1) and (2). This depends, additionally, on the discount rates of the players, $\frac{1-\alpha_i}{\alpha_i}$, not being too large. In particular, existence is assured if

$$\frac{1-\alpha_i}{\alpha_i} < \frac{\pi_i(s^*) - \pi_i(s^c)}{\phi_i(\bar{s}_i^*) - \pi_i(s^c)}.$$

Thus, the following proposition is established:

Proposition 4. *If A1-A9 are true, then a supergame strategy, σ^*, which satisfies* (1) *and* (2) *exists and is, in addition, a non-cooperative equilibrium when*

$$\frac{1-\alpha_i}{\alpha_i} < \frac{\pi_i(s^*) - \pi_i(s^c)}{\phi_i(\bar{s}_i^*) - \pi_i(s^c)}, \quad i = 1, \ldots, n.$$

When σ^* is a non-cooperative equilibrium it might be called the " balanced temptation solution ", for its characteristic (apart from being both Pareto optimal and a non-cooperative equilibrium) is that the ratio of short term gain from maximizing against $\bar{s}_i^*$ to the loss per period of having done so is identical for all players. I.e.:

$$\frac{\pi_i(\bar{s}_i^*, t_i) - \pi_i(s^*)}{\pi_i(s^*) - \pi_i(s^c)} = \frac{\pi_j(\bar{s}_j^*, t_j) - \pi_j(s^*)}{\pi_j(s^*) - \pi_j(s^c)} \text{ for all } i \text{ and } j.$$

An equivalent statement is that σ^* is defined so that $\alpha_i(s^*) = \alpha_j(s^*)$, for all i and j. That is, the discount parameter which makes the ith player indifferent between choosing σ_i^* and choosing $(t_i, s_i^c, s_i^c \ldots)$ against the σ_j^* is the same for all players.

VI. THE RELAXATION OF ASSUMPTIONS

The first assumptions to be dropped are those made at the beginning of section IV: (A6), all constituent games of the supergame are identical; (A7), the discount parameters are the same in all periods; (A8), the basic game has only one non-cooperative equilibrium; and (A9) the non-cooperative equilibrium is not Pareto optimal.

Taking (A9) first, it is immediate that if $\pi(s^c) \in H^*$, then it is the only element of H^*. By default, the supergame equilibrium strategy would be for each player to always choose $s_i^c (i = 1, \ldots, n)$. Relaxing the remaining assumptions, let S_{it} be the strategy set of the ith player in the ordinary game of period t, let $C_t \subset S_t = S_{it} \times \ldots \times S_{nt}$ be the set of non-cooperative equilibria of the ordinary game of period t, and let α_{it} be the present value of the discount parameter of the ith firm in period t. That is, if the one period discount rates are $r_{i1}, \ldots, r_{it}, \ldots$, then

$$\alpha_{it} = \prod_{\tau=1}^{t-1} \frac{1}{1+r_{i\tau}} = \frac{\alpha_{i,t-1}}{1+r_{i,t}}, \quad t = 2, 3, \ldots,$$

$$\alpha_{i1} = 1.$$

If $C = C_1 \times C_2 \times \ldots$, then $c = (c_1, c_2, \ldots) \in C$ is an infinite sequence of ordinary game non-cooperative equilibria, where c_t is a non-cooperative equilibrium in the game described by (S_t, π_t). Proposition 3 proves a result about ordinary games: if $c_t \in C_t$, then the set of points $p_t \in P_t(c_t)$ such that

$$\frac{\phi_{it}(\bar{p}_{it}) - \pi_{it}(c_t)}{\pi_{it}(p_t) - \pi_{it}(c_t)} = \frac{\phi_{jt}(\bar{p}_{jt}) - \pi_{jt}(c_t)}{\pi_{jt}(p_t) - \pi_{jt}(c_t)}, \quad i, j = 1, \ldots, n; \; t = 1, 2, \ldots, \qquad \ldots(3)$$

$$\pi_t(p_t) \in H_t^*, \; \pi_t(p_t) \geqq \pi_t(c_t), \qquad t = 1, 2, \ldots, \qquad \ldots(4)$$

is not empty. The symbols π_{it} and ϕ_{it} are defined as before, except that they are in relation to the game of the tth period.

Let t_{it} be defined as follows:

$$\pi_{it}(\bar{p}_{it}, t_{it}) = \max_{s_{it} \in S_{it}} \pi_i(\bar{p}_{it}, s_{it}).$$

Let $p \in P(c) = P_1(c_1) \times \ldots \times P_t(c_t) \times \ldots$ be a sequence of points satisfying conditions (3) and (4), above. In relation to a given $c \in C$ and $p \in P(c)$, the supergame strategy $\sigma_i(c, p)$ is defined for the ith player:

$$s_{i1} = p_{i1}, \qquad \ldots(5)$$

$$s_{it} = p_{it} \text{ if } s_{j\tau} = p_{j\tau}, \quad j = 1, \ldots, n; \; \tau = 1, \ldots, t-1, \; t \geqq 2, \qquad \ldots(6)$$

$$s_{it} = c_{it} \text{ otherwise.} \qquad \ldots(7)$$

$\sigma(c, p) = [\sigma_1(c, p), \ldots, \sigma_n(c, p)]$ is a non-cooperative equilibrium for the supergame if:

$$\sum_{\tau=t}^{\infty} \alpha_{i\tau}\pi_{it}(p_t) > \alpha_{it}\pi_{it}(\bar{p}_{it}, t_{it}) + \sum_{\tau=t+1}^{\infty} \alpha_{i\tau}\pi_{i\tau}(c_\tau), \quad i = 1, \ldots, n, \quad t = 1, 2, \ldots. \qquad \ldots(8)$$

If these conditions are met, the actual moves chosen will be $p \; (= p_1, p_2, \ldots)$. Here it must be true that no player in any period finds it more profitable to maximize against $\bar{p}_{it}$ and see the future moves be $c_{t+1}, \ldots$. Of course, this was true previously; however, when the same game is repeated in each period and discount rates are invariant over time, it is either never profitable to choose t_i, or most profitable to do so in the first period of the supergame.

If $p_t = c_t$ for all but a finite number of time periods, σ cannot be an equilibrium, and, if that were true for all $c \in C$, the only supergame non-cooperative equilibria would be strategies in which basic game non-cooperative equilibria were repeated.

Thus Proposition 4 is now extended to supergames, satisfying only A1-A5:

Proposition 5. *If A1-A5 are true, then a supergame strategy satisfying* (3)-(7) *exists and is a non-cooperative equilibrium for the supergame if* (8) *is satisfied.*

A4 might be weakened to say that for given $c_t \in C_t$ and $(\rho_1, \ldots, \rho_n)$ exactly one member of the family of vectors $[k(\rho_1, \ldots, \rho_n) + \pi_t(c_t)]$, $\rho_i \geqq 0$, $i = 1, \ldots, n$, $\Sigma\rho_i = 1$, $k \geqq 0$ coincides with a point on the payoff possibility frontier. Thus a surface such as is found in Fig. 2 would be possible. Proposition 3 is still valid. There will be at least one point on the profit frontier, in the segment from a to b. which will map into itself. Such a point provides the basis for a non-cooperative equilibrium which satisfies axioms α4-α6. Pareto optimality cannot be guaranteed. Now two possibilities emerge. (*a*) Do not require Pareto optimality of the solution, merely require that it lie on the frontier. (*b*) If the solution found by (*a*), preceding, is not Pareto optimal, substitute for it the nearest Pareto optimal point which has a larger payoff to each player.

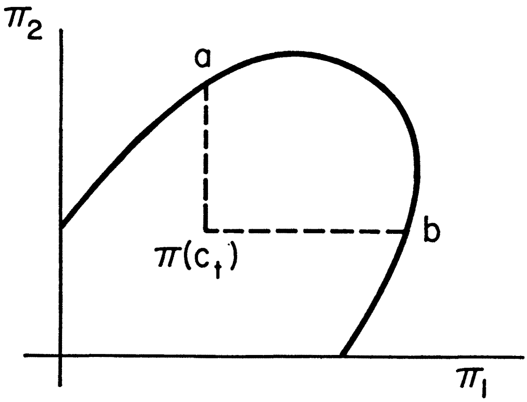

FIGURE 2

VII. COMMENTS ON ECONOMIC APPLICATIONS

A promising area of application for the equilibrium concept developed here is to the theory of oligopoly. With the game interpreted as an oligopoly, the Cournot, or ordinary game non-cooperative, equilibrium is not a Pareto optimal point. Considerable dissatisfaction has been voiced over the years with this equilibrium as a viable outcome in oligopoly. Even though out and out explicit collusion is difficult in a nation having anti-trust legislation, because agreements are not legally binding and even meetings to attempt agreement may be illegal; still it seems unsatisfactory for firms to achieve only the profits of the Cournot point when each firm must realize more can be simultaneously obtained by each.

This line of argument often leads to something called " tacit collusion " under which firms are presumed to act as if they colluded. How they do this is not entirely clear, though one explanation is that their market moves are interpretable as messages. They converse in a code, as it were. Another explanation is that the " tacit collusion " is spontaneous. Everyone is so aware of the shortsightedness of Cournot behaviour that they simply behave better.

Yet, despite these misgivings, the Cournot solution has never been entirely in disrepute. It remains, neither wholeheartedly accepted, nor firmly rejected. No doubt this is because no acceptable alternative has been proposed, and because a non-cooperative equilibrium possesses attractive properties which are hard to entirely forego.

The equilibrium presented here is a sort of reconciliation. It provides an equilibrium which is both Pareto optimal and a non-cooperative equilibrium. Thus, it is possible to see the firm as selfish, willing to make any alteration in its behaviour which will increase its (discounted) profits, and, at the same time, all firms are jointly earning a Pareto optimal vector of profits. They are neither foregoing profit, nor behaving in a way which exposes firms to being " double-crossed ".

In the preceding section, it was found that there may be many basic game non-cooperative equilibria and, for each such point, many Pareto optimal points which could form part of a supergame non-cooperative equilibrium. Were this so, it would be impossible to choose one supergame equilibrium and regard it as a " natural " game solution, i.e., a particular set of strategies one should expect the players to adopt. It is possible, however, that in application to oligopoly, the additional knowledge of the structure of the game may be such as to guarantee existence of only one equilibrium. A unique equilibrium would be a natural solution. Likewise, the difficulty that Proposition 3 could lead to a point on the profit frontier which is not Pareto optimal may turn out impossible in oligopoly models.

The implications of the supergame equilibrium for oligopoly will be explored in more detail in a subsequent paper.

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